Analytical study of chemically reactive Rivlin-Ericksen viscoelastic fluid in a circular tube

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Abstract: This problem deals with the influence of chemically reactive Rivlin-Ericksen viscoelastic fluid in a circular tube with no thermal convection. The fluid starts heat generation because of its reactive nature of chemically viscoelastic fluid which set up free convection currents inside the tube. The governing equations are modelled using the fully developed flow conditions. Analytical algorithm based on the modefied homotopy perturbation method (HPM), incorporating the He's polynomial and combined with the Laplace transform is implemented in time and space with the second grade constitutive model for the viscoelastic liquids. Explicit analytical expressions for the transient state as well as the steady state for velocity field and temperature field have been derived. These solutions are written as the sum between the permanent solutions and the transient solutions. The algorithm is validated against the classical solution of this problem for reactive viscous fluid results. The nature of the wall shear stress and Nusselt number engendered due to the flow are determined. The results also indicate that it takes longer to attain steady-state in the case of molten polymer than water and air.

Key words: Reactive viscoelastic fluid, Transient free convection, Vertical tube, Analytical solution

1 Introduction

In view of the increasing industrial importance of non-Newtonian fluids, the Rivlin-Erickson fluid was examine by Baoku et al. (2013) , Kumar et al. (2013) , Hayat et al. (2013) , Khani (2009) and Olajuwon (2011) .

Transient free convection flow of reactive are extremely useful in the design and operations of many engineering equipments. It has therefore been the subject of many detailed, mostly numerical studies for different flow configurations. Most of the interest in this subject is due to its applications, for instance, in the design of cooling systems for electronic devices and in the field of solar energy collection. Some of the published papers on this topic, such as Makinde (2005) investigated exothermic explosions inside a cylinder pipe for viscous fluid under Biomolecular Arrhenius and sensitized reactive rates neglecting the intake of the materia. Analytical solutions are obtained for the governing nonlinear boundary-value problem using perturbation technique together with a special type of Hermite–Pad e approximants. Makinde (2008) analyzed reactive viscous fluid in a horizontal channel with sliding wall. He obtained aproximate solutions for the governing nonlinear boundary value problem using regular perturbation techniques together with a special type of Hermite–Pad ´e approximants. Makinde (2009) investigated the thermal stability of a reactive viscous flow through a horizontal channel embeded in porous medium with convective boundary condition. The Brinkman model is employed and analytical solutions are constructed for the governing nonlinear boundary-value problem using a perturbation technique together with a special type of Hermite-Padé approximants and important properties of the temperature field including bifurcations and thermal criticality are discussed. Basant et al. (2011*a*) analyzed transient free convection flow of reactive viscous fluid in vertical tube. They obtained the transient solution using numerical schemes and steady-state solution by regular perturbation method. The signaficant results from their study are that both velocity and temperature increases with the increase in the value of reactant consuption parameter and dimensionless time until they reach steady-state value. Basant et al. (2011*b*) investigated the transient free convection flow of reactive viscous fluid in vertical channel. The transient solution for the problem was solved using numerical schemes and steady-state solution by regular perturbation method. The results show that it takes longer time to attain steady-state in the case of water than air.

All the above quoted analyses of transient free convection flow of reactive fluid are based on the hypothesis that the fluids are Newtonian. However, because of their fundamental and technological importance, theoretical studies of transient free convection flow of reactive non-Newtonian are very important in several industrial processes; industrial scale lubricants (mostly reactive polymer liquids). Sivaraj and Rushi Kumar (2013) analyzed the chemically responding dusty viscoelastic fluid (Walters liquid model-B) flow within an irregular channel with convective boundary. They solved the combined partial differential equations analytically using perturbation technique.Their results, reveal that the rate profiles of dusty fluid are greater compared to dust contaminants. Makinde et al. (2011) analyzed an unsteady flow of the reactive variable viscosity non-Newtonian fluid via a porous saturated medium with asymmetric convective boundary conditions. They utilized that exothermic chemical responses occur inside the flow system which the asymmetric convective heat exchange using the ambient in the surfaces follow Newton's law of cooling. The model formulation of coupled nonlinear partial differential equations of the problem are derived and then solved numerically using a semi-implicit finite difference scheme. Rundora and Makinde (2013) looked into the results of suction/injection on unsteady reactive variable viscosity non-Newtonian fluid flow inside a channel full of porous medium and convective boundary conditions. The governing flow equations are solved using a semi-implicit finite difference scheme. Results reveals that, the suction injection Reynolds number, the porous medium parameter and also the Prandtl number possess a diminishing impact on the velocity of and transfer in the channel walls.

In this paper, the analytical solutions of unsteady free convection flow of reactive second grade fluid in vertical tube are presented. To the best of authors knowledge, theoretical study of unsteady free convection flow due to reactive nature of second grade fluid in vertical tube have not been studied before and it is the main aim of this paper to study this problem. The He's polynomial incorporated into the homotopy perturbation method combined with the Laplace transform are used to solve the momentum and the energy equations. Flow and heat transfer results for a range of values of the pertinent parameters have been reported. Effects of pertinent parameters, such as the viscoelastic parameter, the suction/injection parameter, the Grashof number, the Reynolds number, the Prandtl number, the wall temperature parameter and the constant pressure gradient on velocity and temperature profiles are shown graphically

2 Description of the problem

We consider the unsteady two dimensional flow of an incompressible second grade viscoelastic reactive fluid in a vertical pipe with walls at $r = \pm a$. Fig. 1 shows the physical configuration. The $z-$axis is taken along the axis of the pipe and $r-$axis is taken perpendicular to the pipe. Initially, for time $t \leq 0$, both the fluid and the pipe are assumed to be at constant temperature θ_0. At time $t > 0$, the reactive nature of the fluid causes heat generation which set up free convection currents inside the pipe. All the fluid properties except density in the buoyancy term are considered as constant and Boussinesq approximation is employed. The constitutive equation for second grade viscoelastic fluids given by Rivlin and Ericksen (1955) was transform into cylindrical coordinate and then employed. Furthermore, symmetric nature of the flow is taken into account and pressure gradient is neglected. The boundary layer equations for the flow under consideration are

$$\rho \frac{\partial w}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\mu \frac{\partial w}{\partial r}\right) + \alpha_1 \frac{\partial}{\partial t}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right] + g\beta\left(\theta - \theta_0\right), \tag{1}$$

$$\rho c_p \frac{\partial \theta}{\partial t} = \kappa \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) + \mu\left(\frac{\partial w}{\partial r}\right)^2 + \alpha_1 \frac{\partial^2 w}{\partial t \partial r}\frac{\partial w}{\partial r} + QC_0 A \exp\left(-\frac{E}{R\theta}\right). \tag{2}$$

where w is the dimensional velocity of the fluid, θ is the dimensional temperature, α_1 is the viscoelastic parameter of second grade, ρ is the fluid density, μ is the coefficient of viscosity, g is the acceleration due to gravity, β is the volumetric coefficient of thermal expansion, c_p is the specific heat capacity at constant pressure, κ is the thermal conductivity of the fluid, Q is the heat reaction parameter, C_0 is the initial concentration of the reactant species, A is the rate constant, E is the activation energy and R is the universal gas constant

Due to symmetrical nature of the flow at $r = 0$, the initial and boundary conditions are

$$\begin{aligned} t &\leq 0: \; w = 0, \quad \theta = \theta_0 \quad \text{for} \quad 0 \leq r \leq a \\ t &> 0: \; \begin{cases} \dfrac{\partial w}{\partial r} = 0 \quad \dfrac{\partial \theta}{\partial r} = 0 \;\; \text{at } r = 0 \\ w = 0 \quad \theta = \theta_0 \text{ at } r = a \end{cases} \end{aligned} \tag{3}$$

Introducing the following dimensionless variables

$$\begin{aligned} w^* &= \frac{w}{U_0}, \quad t^* = \frac{vt}{a^2}, \quad r^* = \frac{r}{a}, \quad \theta^* = \frac{E}{R\theta_0^2}\left(\theta - \theta_0\right), \\ \lambda^* &= \frac{QC_0 AEa^2}{\kappa R\theta_0^2}\exp\left(-\frac{E}{R\theta_0}\right), \quad \epsilon = \frac{R\theta_0}{E}, \quad \alpha^* = \frac{\alpha_1}{\rho a^2}, \\ Gr &= \frac{g\beta R\theta_0^2 a^3}{Ev^2}, \quad \mathrm{Re} = \frac{au_0}{v}. \quad E_c = \frac{EU_0^2}{c_p R\theta_0^2}, \quad \mathrm{Pr} = \frac{\rho v c_p}{\kappa}. \end{aligned} \tag{4}$$

in Eqs. (12) and (13) and shedding the $*$ notation, we have

$$\frac{\partial w}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \alpha \frac{\partial}{\partial t}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right] + \frac{Gr}{\mathrm{Re}}\theta, \tag{5}$$

$$\mathrm{Pr}\frac{\partial \theta}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) + \mathrm{Pr}\, E_c\left[\left(\frac{\partial w}{\partial r}\right)^2 + \alpha \frac{\partial^2 w}{\partial t \partial r}\frac{\partial w}{\partial r}\right] + \lambda \exp\left(\frac{\theta}{1 + \epsilon\theta}\right). \tag{6}$$

with initial and boundary conditions

$$\begin{aligned} t &\leq 0: \; w=0, \quad \theta=0 \quad \text{for} \quad 0\leq r\leq 1 \\ t &> 0: \begin{cases} \dfrac{\partial w}{\partial r}=0 & \dfrac{\partial \theta}{\partial r}=0 \;\; \text{at } r=0 \\ w=0 & \theta=1 \text{ at } r=1 \end{cases} \end{aligned} \tag{7}$$

where α is the second grade parameter, Gr is the Grashof number, Re is the Reynolds number, E_c is the Echert number, Pr is the Prandtl number, λ is the reactant consumption parameter and ϵ is the activation energy parameter.

The function $\exp\left(\dfrac{\theta}{1+\epsilon\theta}\right)$ can be express in power series as

$$\exp\left(\frac{\theta}{1+\epsilon\theta}\right)=1+\theta+\frac{1}{2}(1-2\epsilon)\theta^2+\frac{1}{3}\left(\frac{1}{2}(1-2\epsilon)+3\epsilon^2\right)\theta^3+... \tag{8}$$

It is important to mention that when $\alpha=0$ (Newtonian fluid), Eqs. (5) and (6) reduce to those found by Basant *et al.* [].

3 Analytical solutions

3.1 Modefied homotopy perturbation Laplace transform method

The differential Eqs. (5) and (6) satisfied by the velocity field $w(r,t)$ and temperature field $\theta(r,t)$ are wtritten in form operator E and F:

$$E(w(r,t)) = 0, \tag{9}$$

$$F(\theta(r,t)) = 0. \tag{10}$$

Basing on the homotopy idea (He 2006), the operators E and F are decomposed into a linear part L and a non-linear part N and are written in the form:

$$L_1(w(r,t))+pN_1(w(r,t)) = g_1(r), \tag{11}$$

$$L_2(\theta(r,t))+pN_2(\theta(r,t)) = g_2(r). \tag{12}$$

where $L_j(j=1,2)$ are linear operator, N_j $(j=1,2)$ are non-linear operator, g_j $(j=1,2)$ are known function

Focusing on the linear operators L_j in Eqs. (11) and (12) the concept of the homotopy perturbation method with embedding parameter p are used to generate a series expansion for $L_1(w(x,t))$ and $L_2(\theta(r,t))$ as follow:

$$L_1(w(r,t)) = L\left(\sum_{i=0}^{\infty} p^i w_i\right), \tag{13}$$

$$L_2(\theta(r,t)) = L\left(\sum_{i=0}^{\infty} p^i \theta_i\right). \tag{14}$$

Switching to the nonlinear operator N_j in Eqs. (11) and (12) we use He's polynomial, H_n, as follow:

$$N_1(w(r,t)) = \sum_{n=0}^{\infty} p^n H_n(w), \tag{15}$$

$$N_2(\theta(r,t)) = \sum_{n=0}^{\infty} p^n H_n(\theta). \tag{16}$$

where the He's polynomial (Ghorbani 2009), H_n, is defined as:

$$H_n = \frac{1}{n!}\frac{d^n}{dp^n} N\left(\sum_{i=0}^{n} p^i w_i\right)_{p=0}. \tag{17}$$

Substituting Eqs. (13) – (16) into Eqs. (11) and (12) we obtain:

$$L_1\left(\sum_{i=0}^{\infty} p^i w_i\right) + \sum_{i=0}^{\infty} p^{i+1} H_i(w) = g_1(r), \tag{18}$$

$$L_2\left(\sum_{i=0}^{\infty} p^i \theta_i\right) + \sum_{i=0}^{\infty} p^{i+1} H_i(\theta) = g_2(r). \tag{19}$$

Taking the Laplace transform from both sides of Eqs. (18) and (19) we get:

$$\mathcal{L}\left\{L_1\left(\sum_{i=0}^{\infty} p^i w_i\right)\right\} + \mathcal{L}\left\{\sum_{i=0}^{\infty} p^{i+1} H_i(w)\right\} = \mathcal{L}(g_1(r)), \tag{20}$$

$$\mathcal{L}\left\{L_2\left(\sum_{i=0}^{\infty} p^i \theta_i\right)\right\} + \mathcal{L}\left\{\sum_{i=0}^{\infty} p^{i+1} H_i(\theta)\right\} = \mathcal{L}(g_2(r)). \tag{21}$$

Eqs. (20) and (21) can be rewritten in the following form:

$$\sum_{i=0}^{\infty} p^i \mathcal{L}\{L(w_i)\} + \sum_{i=0}^{\infty} p^{i+1} \mathcal{L}\{H_i(w)\} = \mathcal{L}\{g_1\}, \tag{22}$$

$$\sum_{i=0}^{\infty} p^i \mathcal{L}\{L(\theta_i)\} + \sum_{i=0}^{\infty} p^{i+1} \mathcal{L}\{H_i(\theta)\} = \mathcal{L}\{g_2\}. \tag{23}$$

Using Eqs. (22) and (23) we introduce the recursive relation:

$$\mathcal{L}\{L_1(w_0)\} = \mathcal{L}\{g_1\}, \tag{24}$$

$$\mathcal{L}\{L(\theta_0)\} = \mathcal{L}\{g_2\}. \tag{25}$$

$$\sum_{i=1}^{\infty} p^i \mathcal{L}\left\{L\left(w_i\right)\right\} + \sum_{i=0}^{\infty} p^{i+1} \mathcal{L}\left\{H_i\left(w\right)\right\} = 0. \tag{26}$$

$$\sum_{i=1}^{\infty} p^i \mathcal{L}\left\{L\left(\theta_i\right)\right\} + \sum_{i=0}^{\infty} p^{i+1} \mathcal{L}\left\{H_i\left(\theta\right)\right\} = 0. \tag{27}$$

3.2 Implementation of modefied HPM

The Laplace transform of Eqs. (5) and (6) are

$$\frac{\partial^2 \bar{w}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}}{\partial r} + \frac{Gr}{\mathrm{Re}}\bar{\theta} = -s\alpha\left(\frac{\partial^2 \bar{w}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}}{\partial r}\right) + s\bar{w}, \tag{28}$$

$$\begin{aligned}\frac{\partial^2 \bar{\theta}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\theta}}{\partial r} &= -\Pr E_c\left[\left(1+s\alpha\right)\left(\frac{\partial \bar{w}}{\partial r}\right)^2\right] - \lambda\left[\frac{1}{s} + \bar{\theta} + \frac{1}{2}\left(1-2\epsilon\right)\bar{\theta}^2\right] \\ &\quad + s\Pr\bar{\theta}.\end{aligned} \tag{29}$$

where $\bar{w}\left(s,r\right) = \int_0^{\infty} w\left(r,t\right)\exp\left(-st\right)dt$ and $\bar{\theta}\left(s,r\right) = \int_0^{\infty} \theta\left(r,t\right)\exp\left(-st\right)dt$

Substituting the recursive equations, Eqs. (26) and (27), into Eqs. (28) and (29) lead to the following equations:

$$\sum_{n=0}^{\infty} p^n\left(\frac{\partial^2 \bar{w}_n}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}_n}{\partial r} + \frac{Gr}{\mathrm{Re}}\bar{\theta}_n\right) = \sum_{n=0}^{\infty} p^{n+1}\left\{-s\alpha\left(\frac{\partial^2 \bar{w}_n}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}_n}{\partial r}\right) + s\bar{w}_n\right\}, \tag{30}$$

$$\sum_{n=0}^{\infty} p^n\left(\frac{\partial^2 \bar{\theta}_n}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\theta}_n}{\partial r}\right) = -\lambda\frac{1}{s} + \sum_{n=0}^{\infty} p^{n+1}\left\{\begin{array}{c} -\Pr E_c\left[\left(1+s\alpha\right)\left(\frac{\partial \bar{w}}{\partial r}\right)^2\right] \\ -\lambda\left[\bar{\theta} + \frac{1}{2}\left(1-2\epsilon\right)\bar{\theta}_n^2\right] + s\Pr\bar{\theta}_n \end{array}\right\} \tag{31}$$

The recursive equations deduce from Eqs. (30) and (31) could be written as follows:

$$\begin{aligned}
\frac{\partial^2 \bar{w}_0}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}_0}{\partial r} + \frac{Gr}{\mathrm{Re}}\bar{\theta}_0 &= 0 \\
\frac{\partial w_0\left(0,s\right)}{\partial r} &= 0, \quad w_0\left(1,s\right) = 0, \\
\frac{\partial^2 \bar{w}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}_1}{\partial r} + \frac{Gr}{\mathrm{Re}}\bar{\theta}_1 &= -s\alpha\left(\frac{\partial^2 \bar{w}_0}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}_0}{\partial r}\right) + s\bar{w}_0, \\
\frac{\partial w_1\left(0,s\right)}{\partial r} &= 0, \quad w_1\left(1,s\right) = 0, \\
\frac{\partial^2 \bar{w}_2}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}_2}{\partial r} + \frac{Gr}{\mathrm{Re}}\bar{\theta}_2 &= -s\alpha\left(\frac{\partial^2 \bar{w}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}_1}{\partial r}\right) + s\bar{w}_1, \\
\frac{\partial w_1\left(0,s\right)}{\partial r} &= 0, \quad w_1\left(1,s\right) = 0.
\end{aligned} \tag{32}$$

and

$$
\begin{aligned}
\frac{\partial^2 \bar{\theta}_0}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\theta}_0}{\partial r} &= -\frac{\lambda}{s}, \\
& \frac{\partial \theta_0\left(0,s\right)}{\partial r} = 0, \quad \theta_0\left(1,s\right) = 0 \\
\frac{\partial^2 \bar{\theta}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\theta}_1}{\partial r} &= -\Pr E_c\left[\left(1+\alpha s\right)\left(\frac{\partial \bar{w}_0}{\partial r}\right)^2\right] - \lambda\left[\bar{\theta}_0 + \frac{1}{2}\left(1-2\epsilon\right)\bar{\theta}_0^2\right] + s\Pr\bar{\theta}_0 \\
& \frac{\partial \theta_1\left(0,s\right)}{\partial r} = 0, \quad \theta_1\left(1,s\right) = 0, \\
\frac{\partial^2 \bar{\theta}_2}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\theta}_2}{\partial r} &= -\Pr E_c\left[2\left(1+\alpha s\right)\frac{\partial \bar{w}_0}{\partial r}\frac{\partial \bar{w}_1}{\partial r}\right] - \lambda\left[\bar{\theta}_1 + \frac{1}{2}\left[2\left(1-2\epsilon\right)\right]\bar{\theta}_0\bar{\theta}_1\right] + s\Pr\bar{\theta}_1 \\
& \frac{\partial \theta_2\left(0,s\right)}{\partial r} = 0, \quad \theta_2\left(1,s\right) = 0.
\end{aligned}
\tag{33}
$$

The solution of the recursive equation, Eqs. (32) and (33), could be expressed by:

$$
\begin{aligned}
\bar{w}_0(r,s) &= \left[\frac{1}{64}\frac{Gr}{Re}\left(3-4r^2+r^4\right)\frac{1}{s}\right]\lambda, \\
\bar{w}_1(r,s) &= \left[\frac{1}{2304}\frac{Gr}{Re}\left(-1+r^2\right)\left(19+r^4+\Pr\left(19-8r^2+r^4\right)\right.\right. \\
&\left.\left. +108\alpha-4r^2\left(2+9\alpha\right)\right)\right]\lambda \\
&+\left[\frac{1}{14745600}\frac{Gr}{Re^3}\left(-1+r^2\right)\left(-6400\left(19-8r^2+r^4\right)Re^2 s\right.\right. \\
&\left.\left. +EcGr^2Re\left(-1916+309r^2+309r^4-91r^6+9r^8\right)\left(\frac{1+s\alpha}{s^2}\right)\right)\right]\lambda^2 \\
&+\left[\frac{1}{73728}\frac{Gr}{Re}\left(59-88r^2+36r^4-8r^6+r^8\right)\frac{(-1+2\epsilon)}{s^2}\right]\lambda^3, \\
\bar{w}_2(r,s) &= \left[\frac{1}{147456}\frac{Gr}{Re}\left(-1+r^2\right)s\left(-211+r^6+Pr^2\left(-211+93r^2-15r^4+r^6\right)\right.\right. \\
& -2432\alpha-6912\alpha^2-r^4\left(15+128\alpha\right)+r^2\left(93+1024\alpha+2304\alpha^2\right) \\
&\left.\left. +\Pr\left(-211+r^6-1216\alpha-r^4\left(15+64\alpha\right)+r^2\left(93+512\alpha\right)\right)\right)\right]\lambda \\
&+\left[\frac{1}{707788800}\frac{1}{Re^3}\frac{1}{s}\left(Gr\left(-1+r^2\right)\left(-4800Re^2 s\left(-211+r^6\right.\right.\right.\right. \\
& +2\Pr\left(-211+98r^2-15r^4+r^6\right)-1216\alpha-r^4\left(15+64\alpha\right) \\
& \left.+r^2\left(93+512\alpha\right)\right)+EcGr^2\Pr\left(1+s\alpha\right)\left(19r^{10}+\Pr\left(49024-11792r^2\right.\right. \\
& \left.-5117r^4+2083r^6-317r^8+19r^{10}\right)-r^8\left(317+1296\alpha\right) \\
& -16r^2\left(737+2781\alpha\right)+64\left(766+4311\alpha\right)+r^6\left(2083+13104\alpha\right) \\
&\left.\left.\left.\left. -r^4\left(5117+44496\alpha\right)\right)\right)\right)\right]\lambda^2 \\
&+\left[\frac{1}{26635508121600}\frac{1}{Re^5}\frac{1}{s^3}\left(Gr\left(-1+r^2\right)\left(-37632EcGr^2\Pr\left(49024\right.\right.\right.\right. \\
& \left.-11792r^2-5117r^4+2083r^6-317r^8+19r^{10}\right)Re^2 s\left(1+s\alpha\right) \\
& +EcGr^4\left(-19149039+2938641r^2+2938641r^4-550159r^6-114059r^8\right. \\
& \left.+55285r^{10}-7435r^{12}+405r^{14}\right)+\left(\Pr+\Pr s\alpha\right)^2-1806336Re^4s^2\left(19073\right. \\
& -11800\alpha+4054\epsilon+23600\alpha\epsilon+\Pr\left(6186-3039r^2+761r^4\right. \\
& \left.-139r^6+11r^8\right)\left(-1+2\epsilon\right)+r^8\left(-2+4\epsilon\right)-r^6\left(77+46\varepsilon+200\alpha\left(-1+2\varepsilon\right)\right) \\
&\left.\left.\left.\left. +r^4\left(1323+354\epsilon+1400\alpha\left(-1+2\epsilon\right)\right)-r^2\left(8377+1846\epsilon+5800\alpha\left(-1+2\epsilon\right)\right)\right)\right)\right)\right]\lambda^3 \\
&-\left[\frac{1}{416179814400}\frac{1}{Re^3}\frac{1}{s^3}Gr\left(-1+r^2\right)\left(-28224\left(6186-3039r^2+761r^4\right.\right.\right. \\
& \left.-139r^6+11r^8\right)Re^2 s+EcGr^2\Pr\left(-3698928+1160304r^2+179079r^4-143096r^6\right. \\
&\left.\left.\left. +43104r^8-6288r^{10}+425r^{12}\right)\left(1+s\alpha\right)\right)\left(-1+2\epsilon\right)\right]\lambda^4 \\
&+\left[\frac{1}{530841600}\frac{1}{Re}\frac{1}{s^3}Gr\left(14529-22113r^2+9900r^4-2900r^6+675r^8\right.\right. \\
&\left.\left. -99r^{10}+8r^{12}\right)\left(1-2\epsilon\right)^2\right]\lambda^5.
\end{aligned}
\tag{34}
$$

and

$$\begin{aligned}
\bar{\theta}_0(r,s) &= \left[\frac{1}{4s}\left(1-r^2\right)\right]\lambda,\\
\bar{\theta}_1(r,s) &= \left[-\frac{1}{16}\,\mathrm{Pr}\left(3-4r^2+r^4\right)\right]\lambda\\
&\quad -\left[\frac{1}{147456Re^2}\left(-1+r^2\right)\left(-2304\left(-3+r^2\right)Re^2s\right.\right.\\
&\quad \left.\left.+EcGr^2\,\mathrm{Pr}\left(89+89r^2-55r^4+9r^6\right)\frac{(1+s\alpha)}{s^2}\right)\right]\lambda^2\\
&\quad +\left[\frac{1}{2304}\left(-11+18r^2-9r^4+2r^6\right)\frac{(-1+2\epsilon)}{s^2}\right]\lambda^3,\\
\bar{\theta}_2(r,s) &= \left[-\frac{1}{2304}Pr^2\left(-19+27r^2-9r^4+r^6\right)s\right]\lambda\\
&\quad -\left[\frac{\mathrm{Pr}}{14745600\,\mathrm{Re}^2}\frac{1}{s}\left(-1+r^2\right)\left(-12800\left(19-8r^2+r^4\right)Re^2s\right.\right.\\
&\quad +EcGr^2\left(1+s\alpha\right)\left(\mathrm{Pr}\left(-5068-2843r^2+2557r^4-648r^6+57r^8\right)\right.\\
&\quad +8\left(-394+6r^8-2225\alpha-3r^6\left(23+75\alpha\right)+r^4\left(281+1375\alpha\right)\right.\\
&\quad \left.\left.\left.\left.-r^2\left(394+2225\alpha\right)\right)\right)\right)\right]\lambda^2\\
&\quad -\left[\frac{1}{208089907200Re^4}\frac{1}{s^3}\left(-1+r^2\right)\left(-14112EcGr^2\,\mathrm{Pr}\left(-5068-2843r^2\right.\right.\right.\\
&\quad \left.+2557r^4-643r^6+57r^8\right)Re^2s\left(1+s\alpha\right)+5Ec^2Gr^4\left(138048+138048r^2\right.\\
&\quad \left.-58197r^4-14587r^6+11873r^8-2239r^{10}+162r^{12}\right)\left(\mathrm{Pr}+\mathrm{Pr}\,s\alpha\right)^2\\
&\quad -1411200Re^4s^2\left(-64\left(19-8r^2+r^4\right)+\mathrm{Pr}\left(-369+239r^2-85r^4+11r^6\right)\right.\\
&\quad \left.\left.\left.\left(-1+2\epsilon\right)\right)\right)\right]\lambda^3\\
&\quad +\left[\frac{1}{2123366400Re^2}\frac{1}{s^3}\left(-1+r^2\right)\left(-14400\left(-369+239r^2-85r^4+11r^6\right)Re^2s\right.\right.\\
&\quad +EcGr^2\,\mathrm{Pr}\left(99168+19068r^2-40107r^4+20693r^6-4507r^8+425r^{10}\right)\\
&\quad \left.\left.\left(1+s\alpha\right)\right)\left(-1+2\epsilon\right)\right]\lambda^4\\
&\quad -\left[\frac{1}{14745600}\frac{1}{s^3}\left(-2457+4400r^2-2900r^4+1200r^6\right.\right.\\
&\quad \left.\left.-275r^8+32r^{10}\right)\left(-1+2\epsilon\right)^2\right]\lambda^5
\end{aligned} \tag{35}$$

Using the MATHEMATICA symbolic code, the inverse Laplace transform of Eqs. (34) and

(35) are:

$$
\begin{aligned}
w_0(r,t) &= \left[\frac{1}{64}\frac{Gr}{Re}\left(3-4r^2+r^4\right)\right]\lambda, \\
w_1(r,t) &= \left[\frac{1}{2304}\frac{Gr}{Re}\left(-1+r^2\right)\left(19+r^4+\Pr\left(19-8r^2+r^4\right)\right.\right. \\
&\left.\left.+108\alpha-4r^2(2+9\alpha)\right)\right]\delta(t)\lambda \\
&+\left[\frac{1}{14745600}\frac{Gr}{Re^3}\left(-1+r^2\right)\left(-6400\left(19-8r^2+r^4\right)Re^2\right.\right. \\
&\left.\left.+EcGr^2Re\left(-1916+309r^2+309r^4-91r^6+9r^8\right)(t+\alpha)\right)\right]\lambda^2 \\
&+\left[\frac{1}{73728}\frac{Gr}{Re}\left(59-88r^2+36r^4-8r^6+r^8\right)(-1+2\epsilon)t\right]\lambda^3, \\
w_2(r,t) &= \left[\frac{1}{147456}\frac{Gr}{Re}\left(-1+r^2\right)\left(-211+r^6+Pr^2\left(-211+93r^2-15r^4+r^6\right)\right.\right. \\
&-2432\alpha-6912\alpha^2-r^4(15+128\alpha)+r^2\left(93+1024\alpha+2304\alpha^2\right) \\
&\left.\left.+\Pr\left(-211+r^6-1216\alpha-r^4(15+64\alpha)+r^2(93+512\alpha)\right)\right)\right]\delta'(t)\lambda \\
&+\left[\frac{1}{707788800}\frac{1}{Re^3}\left(Gr\left(-1+r^2\right)\left(-4800Re^2s\left(-211+r^6\right.\right.\right.\right. \\
&+2\Pr\left(-211+98r^2-15r^4+r^6\right)-1216\alpha-r^4(15+64\alpha) \\
&\left.+r^2(93+512\alpha)\right)+EcGr^2\Pr\alpha\left(19r^{10}+\Pr\left(49024-11792r^2\right.\right. \\
&\left.-5117r^4+2083r^6-317r^8+19r^{10}\right)-r^8(317+1296\alpha) \\
&-16r^2(737+2781\alpha)+64(766+4311\alpha)+r^6(2083+13104\alpha) \\
&\left.\left.\left.-r^4(5117+44496\alpha)\right)\right)\delta(t)\right)+EcGr^3\Pr\left(-1+r^2\right)\left(19r^{10}\right. \\
&+\Pr\left(49024-11792r^2-5117r^4+2083r^6-317r^8+19r^{10}-r^8(317+1296\alpha)\right. \\
&-16r^2(737+2781\alpha)+64(766+4311\alpha)+r^6(2083+13104\alpha) \\
&\left.\left.-r^4(5117+44496\alpha)\right)\right]\lambda^2 \\
&+\left[\frac{1}{53271016243200Re^5}\left(Gr\left(-1+r^2\right)\left(-75264EcGr^2\Pr\left(49024\right.\right.\right.\right. \\
&\left.-11792r^2-5117r^4+2083r^6-317r^8+19r^{10}\right)(t+\alpha)Re^2 \\
&+EcGr^4Pr^2\left(-19149039+2938641r^2+2938641r^4-550159r^6-114059r^8\right. \\
&\left.+55285r^{10}-7435r^{12}+405r^{14}\right)+\left(t+4t\alpha+2\alpha^2\right)^2-3612672\,\mathrm{Re}^4\left(19073\right. \\
&-11800\alpha+4054\epsilon+23600\alpha\epsilon+\Pr\left(6186-3039r^2+761r^4\right. \\
&\left.-139r^6+11r^8\right)(-1+2\epsilon)+r^8(-2+4\epsilon)-r^6(77+46\varepsilon+200\alpha(-1+2\varepsilon)) \\
&\left.\left.\left.\left.+r^4(1323+354\epsilon+1400\alpha(-1+2\epsilon))-r^2(8377+1846\epsilon+5800\alpha(-1+2\epsilon))\right)\right)\right)\right]\lambda^3 \\
&-\left[\frac{1}{832359628800Re^3}Gr\left(-1+r^2\right)\left(-56448\left(6186-3039r^2+761r^4\right.\right.\right. \\
&\left.-139r^6+11r^8\right)Re^2+EcGr^2\Pr\left(-3698928+1160304r^2+179079r^4-143096r^6\right. \\
&\left.\left.\left.+43104r^8-6288r^{10}+425r^{12}\right)(t+2\alpha)\right)(-1+2\epsilon)\right]\lambda^4
\end{aligned}
$$

and

$$
\begin{aligned}
\theta_0(r,t) &= \left[\frac{1}{4}\left(1-r^2\right)\right]\lambda, \\
\theta_1(r,t) &= \left[-\frac{1}{16}\Pr\left(3-4r^2+r^4\right)\delta(t)\right]\lambda \\
&-\left[\frac{1}{147456Re^2}\left(-1+r^2\right)\left(-2304\left(-3+r^2\right)Re^2\right.\right. \\
&\left.\left.+EcGr^2\Pr\left(89+89r^2-55r^4+9r^6\right)(t+\alpha)\right)\right]\lambda^2 \\
&+\left[\frac{1}{2304}\left(-11+18r^2-9r^4+2r^6\right)(-1+2\epsilon)\,t\right]\lambda^3 \\
\theta_2(r,t) &= \left[-\frac{1}{2304}Pr^2\left(-19+27r^2-9r^4+r^6\right)\delta'(t)\right]\lambda \\
&+\left[\frac{-1}{14745600Re^2}\Pr\left(-1+r^2\right)\left(-12800\left(19-8r^2+r^4\right)Re^2\right.\right. \\
&-EcGr^2\alpha\left(\Pr\left(5068+2843r^2-2557r^4+648r^6-57r^8\right)\right. \\
&-8\left(-394+6r^8-2225\alpha-3r^6(23+75\alpha)+r^4(281+1375\alpha)\right. \\
&\left.\left.\left.-r^2(394+2225\alpha)\right)\right)\right)\delta(t)-\frac{EcGr^2\Pr}{14745600Re^2}\left(-1+r^2\right) \\
&\left(\Pr\left(-5068-2843r^2+2557r^4-648r^6+57r^8\right)\right. \\
&+8\left(-394+6r^8-2225\alpha-3r^6(23+75\alpha)+r^4(281+1375\alpha)\right. \\
&\left.\left.-r^2(394+2225\alpha)\right)\right]\lambda^2 \\
&-\left[\frac{1}{416179814400\,\mathrm{Re}^4}\left(-1+r^2\right)\left(-28224EcGr^2\Pr\left(-5068-2843r^2\right.\right.\right. \\
&\left.+2557r^4-643r^6+57r^8\right)Re^2(t+\alpha)+5Ec^2Gr^4Pr^2\left(138048+138048r^2\right. \\
&\left.-58197r^4-14587r^6+11873r^8-2239r^{10}+162r^{12}\right)\left(t^2+4t\alpha+2\alpha^2\right) \\
&-2822400\,\mathrm{Re}^4\left(-64\left(19-8r^2+r^4\right)+\Pr\left(-369+239r^2-85r^4+11r^6\right)\right. \\
&\left.\left.(-1+2\epsilon)\right)\right)\Big]\lambda^3 \\
&+\left[\frac{1}{4246732800\,\mathrm{Re}^2}\left(-1+r^2\right)t\left(-2800\left(-369+239r^2-85r^4+11r^6\right)Re^2\right.\right. \\
&+EcGr^2\Pr\left(99168+19068r^2-40107r^4+20693r^6-4507r^8+425r^{10}\right) \\
&\left.\left.(t+2\alpha)\right)(-1+2\epsilon)\right]\lambda^4 \\
&-\left[\frac{1}{29491200}\left(-2457+4400r^2-2900r^4+1200r^6-275r^8+32r^{10}\right)(-1+2\epsilon)^2t^2\right]\lambda^5
\end{aligned} \tag{37}
$$

where $\delta(t)$ denotes Diract delta function and $\delta'(t)$ is the derivative of $\delta(t)$.

Thus, the modified homotopy solution for velocity field and temperature distribution correct up to second order are given by

$$
\begin{aligned}
w(r,t) &= \left[\frac{1}{64}\frac{Gr}{Re}\left(3-4r^2+r^4\right)\right]\lambda \\
&+\left[\frac{1}{2304}\frac{Gr}{Re}\left(-1+r^2\right)\left(19+r^4+\Pr\left(19-8r^2+r^4\right)\right.\right. \\
&\left.\left.+108\alpha-4r^2\left(2+9\alpha\right)\right)\right]\delta(t)\lambda \\
&+\left[\frac{1}{14745600}\frac{Gr}{Re^3}\left(-1+r^2\right)\left(-6400\left(19-8r^2+r^4\right)Re^2\right.\right. \\
&\left.\left.+EcGr^2Re\left(-1916+309r^2+309r^4-91r^6+9r^8\right)(t+\alpha)\right)\right]\lambda^2 \\
&+\left[\frac{1}{73728}\frac{Gr}{Re}\left(59-88r^2+36r^4-8r^6+r^8\right)(-1+2\epsilon)t\right]\lambda^3 \\
&+\left[\frac{1}{147456}\frac{Gr}{Re}\left(-1+r^2\right)\left(-211+r^6+Pr^2\left(-211+93r^2-15r^4+r^6\right)\right.\right. \\
&-2432\alpha-6912\alpha^2-r^4\left(15+128\alpha\right)+r^2\left(93+1024\alpha+2304\alpha^2\right) \\
&\left.\left.+\Pr\left(-211+r^6-1216\alpha-r^4\left(15+64\alpha\right)+r^2\left(93+512\alpha\right)\right)\right)\right]\delta'(t)\lambda \\
&+\left[\frac{1}{707788800}\frac{1}{Re^3}\left(Gr\left(-1+r^2\right)\left(-4800Re^2s\left(-211+r^6\right.\right.\right.\right. \\
&+2\Pr\left(-211+98r^2-15r^4+r^6\right)-1216\alpha-r^4\left(15+64\alpha\right) \\
&\left.+r^2\left(93+512\alpha\right)\right)+EcGr^2\Pr\alpha\left(19r^{10}+\Pr\left(49024-11792r^2\right.\right. \\
&\left.-5117r^4+2083r^6-317r^8+19r^{10}\right)-r^8\left(317+1296\alpha\right) \\
&-16r^2\left(737+2781\alpha\right)+64\left(766+4311\alpha\right)+r^6\left(2083+13104\alpha\right) \\
&\left.\left.\left.-r^4\left(5117+44496\alpha\right)\right)\right)\delta(t)\right)+EcGr^3\Pr\left(-1+r^2\right)\left(19r^{10}\right. \\
&+\Pr\left(49024-11792r^2-5117r^4+2083r^6-317r^8+19r^{10}-r^8\left(317+1296\alpha\right)\right. \\
&-16r^2\left(737+2781\alpha\right)+64\left(766+4311\alpha\right)+r^6\left(2083+13104\alpha\right) \\
&\left.\left.-r^4\left(5117+44496\alpha\right)\right)\right]\lambda^2 \\
&+\left[\frac{1}{53271016243200Re^5}\left(Gr\left(-1+r^2\right)\left(-75264EcGr^2\Pr\left(49024\right.\right.\right.\right. \\
&\left.-11792r^2-5117r^4+2083r^6-317r^8+19r^{10}\right)(t+\alpha)Re^2 \\
&+EcGr^4Pr^2\left(-19149039+2938641r^2+2938641r^4-550159r^6-114059r^8\right. \\
&\left.+55285r^{10}-7435r^{12}+405r^{14}\right)+\left(t+4t\alpha+2\alpha^2\right)^2-3612672\,\mathrm{Re}^4\left(19073\right. \\
&-11800\alpha+4054\epsilon+23600\alpha\epsilon+\Pr\left(6186-3039r^2+761r^4\right. \\
&\left.-139r^6+11r^8\right)(-1+2\epsilon)+r^8(-2+4\epsilon)-r^6\left(77+46\varepsilon+200\alpha(-1+2\varepsilon)\right) \\
&\left.\left.\left.\left.+r^4\left(1323+354\epsilon+1400\alpha(-1+2\epsilon)\right)-r^2\left(8377+1846\epsilon+5800\alpha(-1+2\epsilon)\right)\right)\right)\right)\right]\lambda^3 \\
&-\left[\frac{1}{832359628800Re^3}Gr\left(-1+r^2\right)\left(-56448\left(6186-3039r^2+761r^4\right.\right.\right. \\
&\left.-139r^6+11r^8\right)Re^2+EcGr^2\Pr\left(-3698928+1160304r^2+179079r^4-143096r^6\right. \\
&\left.\left.\left.+43104r^8-6288r^{10}+425r^{12}\right)(t+2\alpha)\right)(-1+2\epsilon)\right]\lambda^4
\end{aligned}
$$

and

$$
\begin{aligned}
\theta(r,t) &= \left[\frac{1}{4}\left(1-r^2\right)\right]\lambda \\
&+\left[-\frac{1}{16}\Pr\left(3-4r^2+r^4\right)\delta(t)\right]\lambda \\
&-\left[\frac{1}{147456Re^2}\left(-1+r^2\right)\left(-2304\left(-3+r^2\right)Re^2\right.\right. \\
&\left.\left.+EcGr^2\Pr\left(89+89r^2-55r^4+9r^6\right)(t+\alpha)\right)\right]\lambda^2 \\
&+\left[\frac{1}{2304}\left(-11+18r^2-9r^4+2r^6\right)(-1+2\epsilon)t\right]\lambda^3 \\
&+\left[-\frac{1}{2304}Pr^2\left(-19+27r^2-9r^4+r^6\right)\delta'(t)\right]\lambda \\
&+\left[\frac{-1}{14745600Re^2}\Pr\left(-1+r^2\right)\left(-12800\left(19-8r^2+r^4\right)Re^2\right.\right. \\
&-EcGr^2\alpha\left(\Pr\left(5068+2843r^2-2557r^4+648r^6-57r^8\right)\right. \\
&-8\left(-394+6r^8-2225\alpha-3r^6(23+75\alpha)+r^4(281+1375\alpha)\right. \\
&\left.\left.\left.-r^2(394+2225\alpha)\right)\right)\right)\delta(t)-\frac{EcGr^2\Pr}{14745600Re^2}\left(-1+r^2\right) \\
&\left(\Pr\left(-5068-2843r^2+2557r^4-648r^6+57r^8\right)\right. \\
&+8\left(-394+6r^8-2225\alpha-3r^6(23+75\alpha)+r^4(281+1375\alpha)\right. \\
&\left.\left.\left.-r^2(394+2225\alpha)\right)\right)\right]\lambda^2 \\
&-\left[\frac{1}{416179814400\,\mathrm{Re}^4}\left(-1+r^2\right)\left(-28224EcGr^2\Pr\left(-5068-2843r^2\right.\right.\right. \\
&\left.+2557r^4-643r^6+57r^8\right)Re^2(t+\alpha)+5Ec^2Gr^4Pr^2\left(138048+138048r^2\right. \\
&\left.-58197r^4-14587r^6+11873r^8-2239r^{10}+162r^{12}\right)\left(t^2+4t\alpha+2\alpha^2\right) \\
&-2822400\,\mathrm{Re}^4\left(-64\left(19-8r^2+r^4\right)+\Pr\left(-369+239r^2-85r^4+11r^6\right)\right. \\
&\left.\left.\left.(-1+2\epsilon)\right)\right)\right]\lambda^3 \\
&+\left[\frac{1}{4246732800\,\mathrm{Re}^2}\left(-1+r^2\right)t\left(-2800\left(-369+239r^2-85r^4+11r^6\right)Re^2\right.\right. \\
&+EcGr^2\Pr\left(99168+19068r^2-40107r^4+20693r^6-4507r^8+425r^{10}\right) \\
&\left.\left.(t+2\alpha)\right)(-1+2\epsilon)\right]\lambda^4 \\
&-\left[\frac{1}{29491200}\left(-2457+4400r^2-2900r^4\right.\right. \\
&\left.\left.+1200r^6-275r^8+32r^{10}\right)(-1+2\epsilon)^2t^2\right]\lambda^5 \qquad (39)
\end{aligned}
$$

3.3 Nusselt number and wall friction

The expressions for Nusselt number and wall friction, which are measures of the heat transfer rate and shear stress at the inner surface of the tube respectively, are presented in the following

$$\begin{aligned}
\tau \;=\; & \lim_{r\to 1}\left(\frac{\partial u}{\partial r}\right)=\left[\frac{Gr}{Re}\left(-\frac{1}{16}+\frac{1}{96}\left(1+\Pr+6\alpha\right)\delta\left(t\right)-\frac{1}{6144}\left(11+11Pr^{2}+128\alpha\right.\right.\right.\\
& \left.\left.\left.+384\alpha^{2}+\Pr\left(11+64\alpha\right)\right)\right)\delta^{'}\left(t\right)\right]\lambda\\
& +\left[\frac{1}{5898240Re^{3}}\left(-61440GrRe^{2}+EcGr^{3}\Pr\left(565+565\Pr-1104t+2208\alpha\right)\right)\right.\\
& \left.+\frac{Gr}{5898240Re^{3}}\left(960Re^{2}\left(11+22\Pr+64\alpha\right)+EcGr^{3}\Pr\alpha\left(565+565\Pr+3312\alpha\right)\right)\delta\left(t\right)\right]\lambda^{2}\\
& +\left[-\frac{1}{10569646080}\frac{Gr}{Re^{5}}\left(1012480EcGr^{2}Pr^{2}\left(t+\alpha\right)+5511Ec^{2}Gr^{4}Pr^{2}\left(t^{2}+4t\alpha+2\alpha^{2}\right)\right)\right.\\
& \left.+86016Re^{4}\left(199+120t-120\alpha+42\epsilon-24t\epsilon+240\alpha\epsilon+63\Pr\left(-1+2\epsilon\right)\right)\right)\Big]\lambda^{3}\\
& +\left[\frac{1}{495452160}\frac{Gr}{Re^{3}}t\left(254016Re^{2}+2935EcGr^{2}\Pr\left(t+2\alpha\right)\right)\left(-1+2\epsilon\right)\right]\lambda^{4}\\
& +\left[-\frac{73}{4423680Re}\left(1-2\epsilon\right)^{2}\right]\lambda^{5} \qquad (40)
\end{aligned}$$

$$\begin{aligned}
Nu \;=\; & \lim_{r\to 1}\left(\frac{\partial \theta}{\partial r}\right)=\left[-\frac{1}{2}+\frac{1}{16}\Pr\delta\left(t\right)-\frac{1}{96}Pr^{2}\delta^{'}\left(t\right)\right]\lambda\\
& +\left[\frac{1}{1228800Re^{2}}\Pr\left(2560Re^{2}+EcGr^{2}\alpha\left(76+99\Pr+440\alpha\right)\right)\delta\left(t\right)\right]\lambda^{2}\\
& +\left[-\frac{1}{495452160Re^{4}}\left(399168EcGr^{2}\Pr\left(t+\alpha\right)+2537Ec^{2}Gr^{4}Pr^{2}\left(t^{2}+4t\alpha+2\alpha^{2}\right)\right.\right.\\
& \left.\left.+80640Re^{4}\left(17\Pr\left(-1+2\epsilon\right)+32\left(2+t-2t\epsilon\right)\right)\right)\right]\lambda^{3}\\
& +\left[\frac{1}{35389440Re^{2}}t\left(97920Re^{2}+1579EcGr^{2}\Pr\left(t+2\alpha\right)\right)\left(-1+2\epsilon\right)\right]\lambda^{4}\\
& +\left[-\frac{7}{81920}t^{2}\left(1-2\epsilon\right)^{2}\right]\lambda^{5}. \qquad (41)
\end{aligned}$$

4 Results and discussion

To study the effects of reactant consumption parameter (λ), the activation energy parameter (ϵ), the Grashof number (Gr), the Reynolds number (Re), the Prandtl number (Pr), the viscoelastic parameter (α) on flow field in the boundary layer region, the numerical values of velocity and temperature profiles, computed from the analytical solution mentioned in Sects. 2 and 3, are displayed graphically versus boundary layer coordinate r in Figs. 1, 2, 3, 4 and 5 for various values of consumption parameter λ, viscoelastic parameter α, Grashof number Gr and for Prandtl number corresponding to non-Newtonian fluid $(\Pr > 7)$ for fixed activation energy ϵ.

The expression for temperature field $\theta(r,t)$, provided given by Eq. (42), are shown graphically in Figs. 2 and 3. Fig. 2 present the effect of λ on θ. It is observed from this figure that as λ increases the temperature increases increases which is physically true because as reaction increases the temperature within the fluid will increase. The effect of $\Pr$ on θ have been shown in Fig. 3. Two working fluids are water $(\Pr = 7)$ and for some dilute polymer $(\Pr = 10, ...50)$. It is evident from this figure that the temperature of water is greater than that of molten polymer. This property is preserved for all t, showing the inverse variation of θ with $\Pr$.

The expression for velocity field $w(r,t)$, given by Eq. (41), are shown graphically in Figs. $4-6$ for small value of time $(t = 0.2)$ and large value of time $(t = 6)$. Effect of λ on w is shown in Fig. 4 as λ increases the fluid velocity increases. Also, the influence of the λ on the velocity field is significant for large values of the time t. Fig. 5 displays the effect of Gr on w. Since Gr signifies the relative effects of thermal buoyancy force to viscous hydrodynamic force in the boundary layer region. It is observed from Fig. 5 that an increase in Gr leads to an increase in fluid velocity in the boundary layer region. This implies that thermal buoyancy force tends to accelerate fluid flow in vertical tube. Figure 6 demonstrates the effects of α on w. It is found from Fig. 6 that the influence of viscoelastic parameter α on the fluid motion is signifficant only for small values of the time t. We can see from Fig. 6 that as α increases the velocity of the fluid decreases continuously inside the tube. From this Figure, we can also compare the velocity of viscoelastic with velocity corresponding to Newtonian fluid $(\alpha = 0)$. For small values of time t, the Newtonian fluid flows faster than viscoelastic fluid. For increasing t, the viscoelastic fluid tend to steady state.

The numerical values of non-dimensional wall shear stress τ, computed from the analytical expression given by Eq. (), are presented in tabular form in Tables 1 and 2 for various values of α, λ and t taking $\Pr = 10$, $Gr = \mathrm{Re} = Ec = 1$ and $\epsilon = 0.01$ while that of Nusselt number Nu, computed from the analytical expression presented given by Eq. (), are displayed in tabular form in Tables 3 and 4 for different values of $\Pr$, λ and t. It is evident from Table 1 that the wall shear stress increases as the time t progresses while it decreases on increasing α which imply that time has tendency to enhance skin friction whereas viscoelastic parameter has reverse effect on it. In Table 2, wall shear stress is tabulated against reactant consumption parameter λ, for different values of time t and $\alpha = 0.4$. It is obseved that as λ increases the wall shear stress increases. This really is physically true because when λ increases the velocity increases and therefore there is high wall shear stress around the boundary. Table 3 show numerical values for Nusselt number for different values of time t and reactant consumtion parameter λ at fixed $\mathrm{Pr} = 10$ and $\epsilon = 0.01$. This table reflect that as time increases the rate of heat transfer on the tube surface increases. Also, It is observed that as λ increases the rates of heat transfer increases. This is due to fact that as λ enhances the fluid temperature increases consequently the temperature gradient increases which lead to a boost in the rate of heat transfer. It is evident from Table 3 that Nusselt number Nu decreases on increasing Prandtl number Pr which implies that thermal diffusion tends to increase rate of heat transfer on the tube. From this table, we can also compare the Nusselt number of viscoelastic fluid with Nusselt number corresponding to Newtonian fluid ($\alpha = 0$). It is noticed that, Nusselt number is higher in case of water ($\mathrm{Pr} = 7$) than dilute polymer ($\mathrm{Pr} = 10,\ 15,\ 20...$) with an increase in consumption parameter λ.It is observed from Table 4 that Nusselt number Nu increase on increasing both time t and consumption parameter λ which implies that there is an enhancement in the rate of heat transfer on tube when both t and λ increases.

5 Conclusions

Analytical solutions for the velocity field, temperature distribution, wall shear stress and Nusselt number are obtained for reactive viscoelastic fluid in a circular tube to analyzed the effects of flow type including heat generation current inside the circular tube. The result shows that the pertinent parameters, such as reactant consumption parameter, λ, viscoelastic parameter, α, activation energy parameter, ϵ, Grashof number, Gr, Prandtl number, Pr, and time, t, are primary factors affecting the fluid flow and heat transfer performance in a circular tube. It is discovered that as λ increases the velocity and temperature gradients increase leading to an increase in the wall shear stress and rate of heat transfer on the circular tube surface.

Reactant consumption parameter tends to accelerate the fluid temperature whereas thermal diffusion have reverse effect on it. Thermal buoyancy force and reactant consumption parameter tends to enhance fluid velocity. Also, fluid velocity is accelerated as time progresses. Fluid velocity is slower in the case viscoelastic fluid than that in case of Newtonian fluid.

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Fig. 1 Schematic representation of the physical model

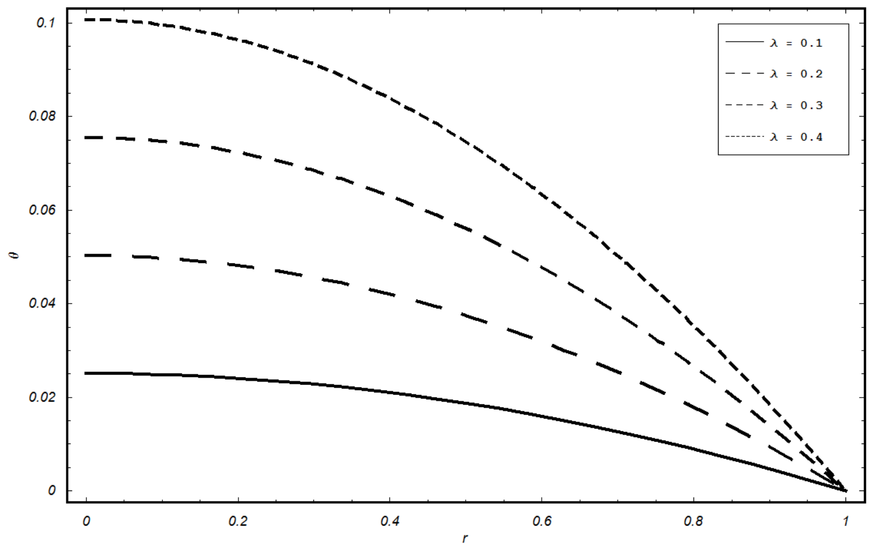


Fig. 2 Effect of the parameter λ on θ when $\alpha = 0.4$, $\epsilon = 0.01$, $\Pr = 10$, $t = 0.2$ and $Gr = Ec = Re = 1$.

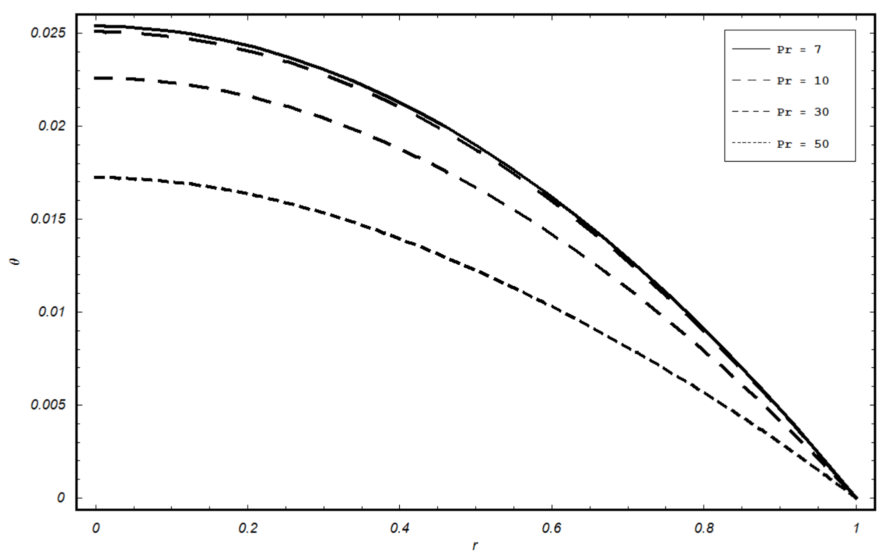
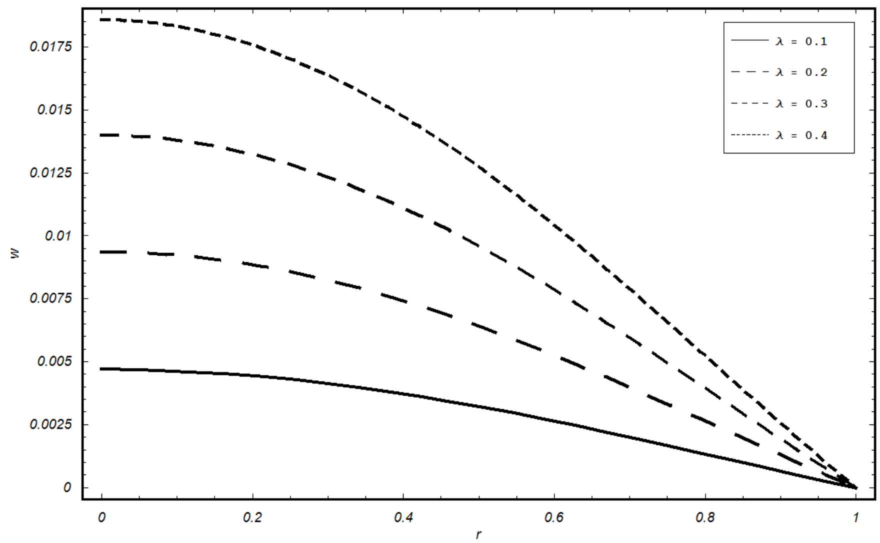
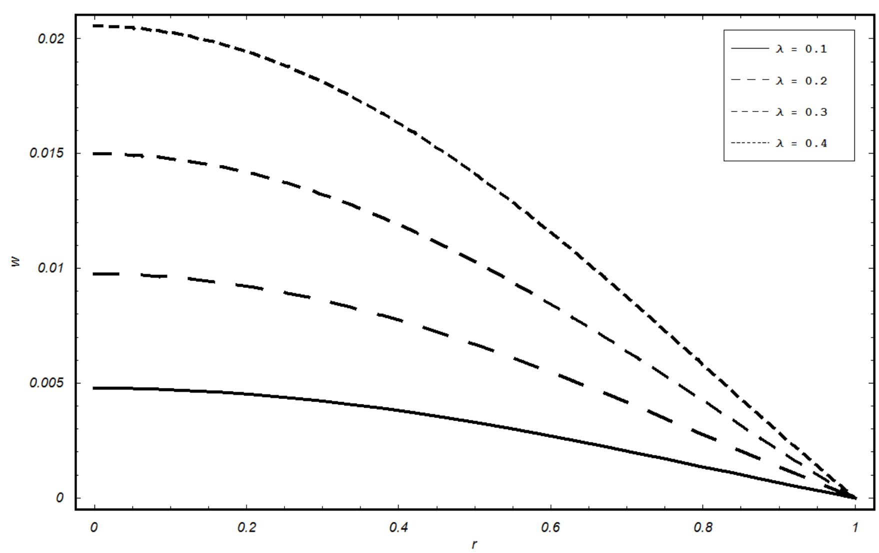


Fig. 3 Effect of the parameter Pr on θ when $\alpha = 0.4$, $\epsilon = 0.01$, $\lambda = 0.2$, $t = 0.2$ and $Gr = Ec = Re = 1$.

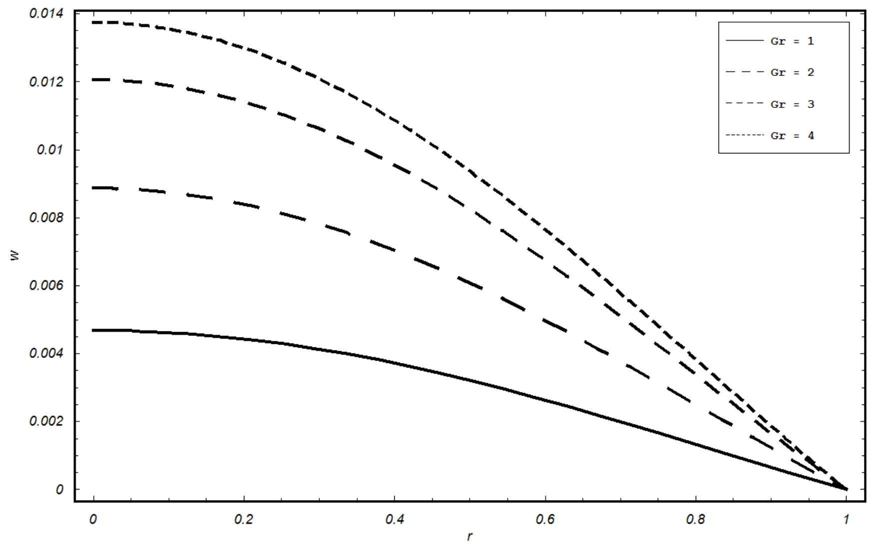


small time ($t = 0.2$)

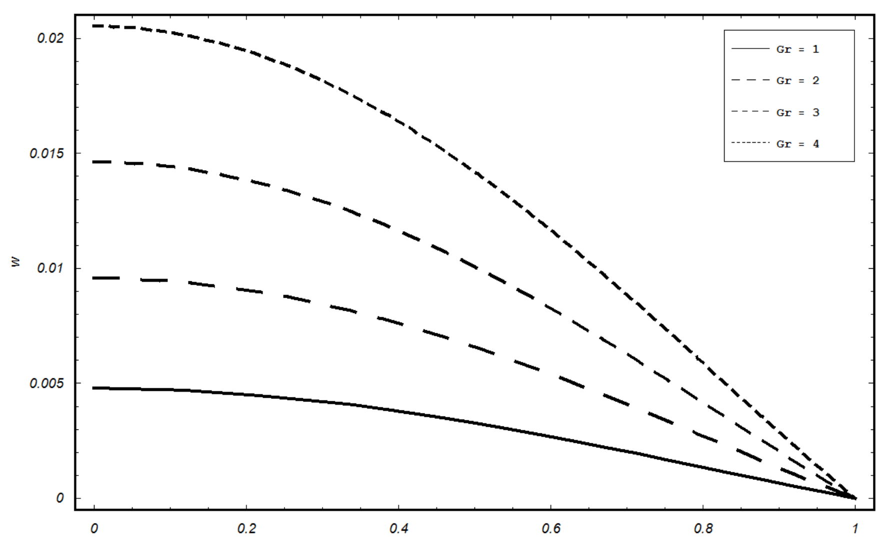


large time ($t = 6$)

Fig. 4 Effect of the parameter λ on w when $\epsilon = 0.01$, $\Pr = 10$, $\alpha = 0.4$ and $Gr = Ec = Re = 1$.



small time ($t = 0.2$)



large time ($t = 6$)

Fig. 5 Effect of the parameter Gr on w when $\epsilon = 0.01$, $\Pr = 10$, $\alpha = 0.4$, $\lambda = 0.1$ and $Ec = Re = 1$.

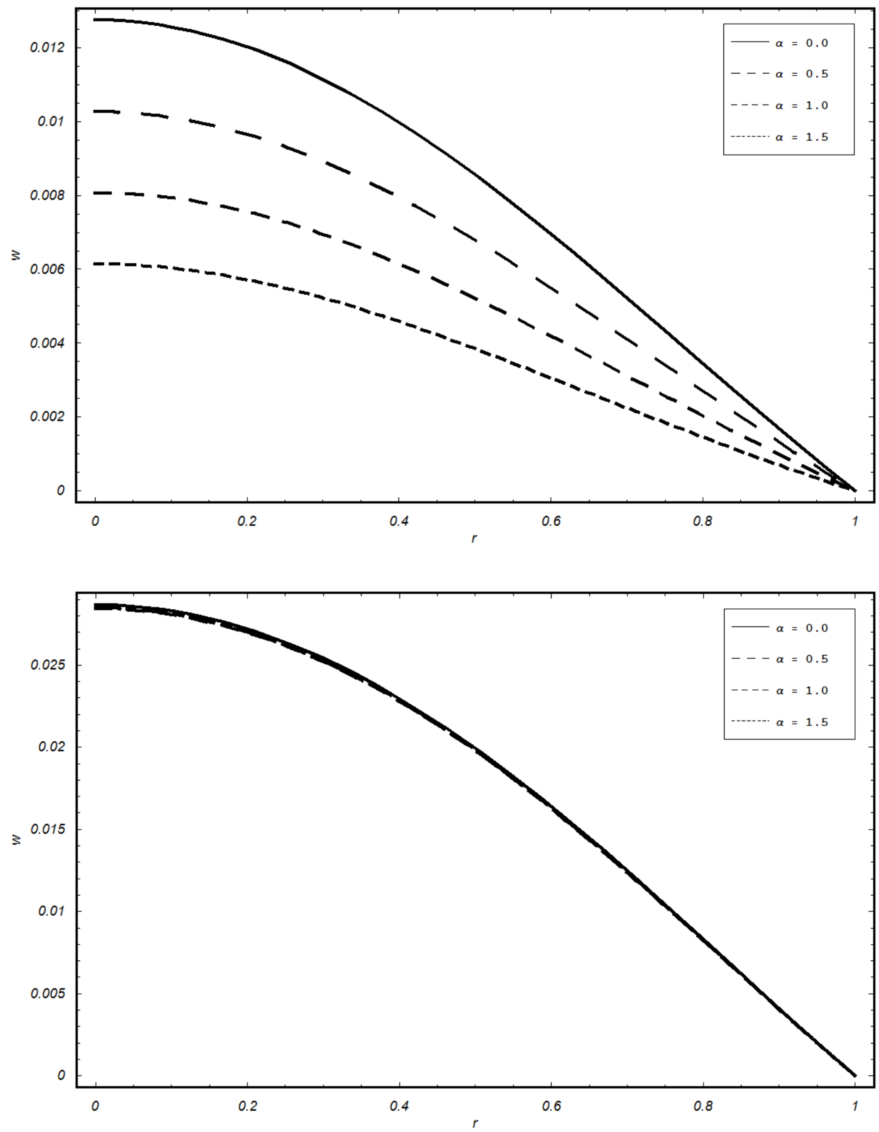


Fig. 6 Effect of the parameter Gr on w when $\epsilon = 0.01$, $\Pr = 10$, $\alpha = 0.4$, $\lambda = 0.1$ and $Ec = Re = 1$.