On two general nonlocal problems of an arbitrary (fractional) orders differential equations

Abd El-Salam Sh. A.(1) and Gaafar F. M.(2)
(1)shrnahmed@yahoo.com
Faculty of Science, Damanhour University, Damanhour, Egypt
(2)fatmagaafar2@yahoo.com
Faculty of Science, Damanhour University, Damanhour, Egypt

Abstract

In this paper, we prove some local and global existence theorems for a fractional orders differential equations with nonlocal conditions, also the uniqueness of the solution will be studied.

Key words: Fractional calculus; fractional order differential equations with nonlocal conditions.

1 Introduction

In this work, we consider an arbitrary (fractional) orders differential equation of the form:

$$\frac{du}{dt} = f(t, D^{\alpha}\, u(t)), \quad \alpha \in (0,\ 1) \tag{1}$$

with the nonlocal conditions

$$I^{\alpha} u(t)|_{t\,=\,\eta} = I^{\alpha}\, u(t)|_{t\,=\,1}, \quad \eta \in (0,\ 1) \tag{2}$$

or

$$t^{1\,-\,\alpha} u(t)|_{t\,=\,\eta} = t^{1\,-\,\alpha}\, u(t)|_{t\,=\,1}, \quad \eta \in (0,\ 1) \tag{3}$$

The nonlocal problems have been intensively studied by many authors, for instance in [4], the authors proved the existence of L_1-solution of the nonlocal boundary value problem

$$\begin{cases} D^{\beta}\, u(t) + f(t, u(\phi(t))) = 0,\ \beta \in (1,2),\ t \in (0,1), \\ \\ I^{\gamma}\, u(t)|_{t=0} = 0, \gamma \in (0,1],\ \alpha\, u(\eta) = u(1),\ 0 < \eta < 1,\ 0 < \alpha\, \eta^{\beta-1} < 1. \end{cases}$$

where the function f satisfies Caratheodory conditions and the growth condition.
And, in [3], the authors proved by using the Banach contraction fixed point theorem, the existence of a unique solution of the fractional-order differential equation:

$${}_{C}D^{\alpha}\, x(t) = c(t)\, f(x(t)) + b(t),$$

with the nonlocal condition:

$$x(0) \ + \ \sum_{k=1}^{m} a_k \ x(t_k) \ = \ x_0,$$

where $x_0 \in \Re$ and $0 < t_1 < t_2 < \cdots < t_m < 1$, and $a_k \neq 0$ for all $k = 1, 2, \cdots, m$.
(Where ${}_C D^\alpha$ is the Caputo derivative).

2 Preliminaries

Define $L_1(I)$ as the class of Lebesgue integrable functions on the interval $I = [a, b]$, where $0 \leq a < b < \infty$ and let $\Gamma(.)$ be the gamma function. Let $C(U, X)$ be The set of all compact operators from the subspace $U \subset X$ into the Banach space X and let $B_r = \{u \in L_1(I) : ||u|| < r, r > 0\}$.

Definition 1.1 The fractional integral of the function $f(.) \in L_1(I)$ of order $\beta \in R^+$ is defined by (see [5] - [8])

$$I_a^\beta \ f(t) \ = \ \int_a^t \frac{(t \ - \ s)^{\beta \ - \ 1}}{\Gamma(\beta)} \ f(s) \ ds.$$

Definition 1.2 The Riemann-Liouville fractional-order derivative of $f(t)$ of order $\alpha \in (0, 1)$ is defined as (see [5] - [8])

$$D_a^\alpha \ f(t) \ = \ \frac{d}{dt} \ I_a^{1 \ - \ \alpha} \ f(t), \quad t \ \in \ [a, b].$$

In this paper, we prove the existence of L_1-solutions for problems (1) - (2) and (1) - (3).Also, we will study the uniqueness of the solution.

Now, let us state the theorems which will be needed in the sequel.

Theorem 2.1 (Rothe Fixed Point Theorem) [1]

Let U be an open and bounded subset of a Banach space E, let $T \in C(\bar{U}, E)$. Then T has a fixed point if the following condition holds

$$T(\partial U) \ \subseteq \ \bar{U}.$$

Theorem 2.2 (Nonlinear alternative of Laray-Schauder type) [1]

Let U be an open subset of a convex set D in a Banach space E. Assume $0 \in U$ and $T \in C(\bar{U}, E)$. Then either

(A1) T has a fixed point in $\bar{U}$, or

(A2) there exists $\gamma \in (0, 1)$ and $x \in \partial U$ such that $x \ = \ \gamma \ Tx$.

Theorem 2.3 (Kolmogorov compactness criterion) [2]

Let $\Omega \subseteq L^p(0,1),\ 1 \leq p < \infty$. If

(i) Ω is bounded in $L^p(0,1)$ and

(ii) $x_h \to x$ as $h \to 0$ uniformly with respect to $x \in \Omega$, then Ω is relatively compact in $L^p(0,1)$, where

$$x_h(t) = \frac{1}{h} \int_t^{t+h} x(s)\, ds.$$

3 Main results

Firstly, we will prove the equivalence of equation (1) with the corresponding Volterra integral equation:

$$y(t) = \frac{u_0\, t^{-\alpha}}{\Gamma(1-\alpha)} + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s, y(s))\, ds, \quad t \in (0,1). \tag{4}$$

Indeed: integrate both sides of (1), we get

$$u(t) - u_0 = I\, f(t,\ D^\alpha u(t)), \tag{5}$$

Now, operating by $I^{1-\alpha}$ on both sides of (5), then

$$I^{1-\alpha} u(t) - I^{1-\alpha} u_0 = I^{2-\alpha} f(t,\ D^\alpha u(t)). \tag{6}$$

Differentiating both sides we get

$$D^\alpha u(t) - \frac{u_0\, t^{-\alpha}}{\Gamma(1-\alpha)} = I^{1-\alpha} f(t,\ D^\alpha u(t)).$$

Take $y(t) = D^\alpha u(t)$, we get (4)
Conversely, operate by I^α on both sides of (6), and differentiate twice we obtain (1).

Now define the operator T as

$$Ty(t) = \frac{u_0\, t^{-\alpha}}{\Gamma(1-\alpha)} + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s, y(s))\, ds, \quad t \in (0,1).$$

To solve equation (4), we must prove that the operator T has a fixed point.

Consider the following assumptions:

(a) $f : (0,1) \times R \to R$ be a function with the following properties:

(i) for each $t \in (0,1), f(t,.)$ is continuous,

(ii) for each $y \in R, f(.,y)$ is measurable,

(iii) there exist two real functions $t \to a(t), t \to b(t)$ such that

$$| \ f(t,y) \ | \ \leq \ a(t) \ + \ b(t) \ | \ y \ |, \quad \text{for each} \quad t \in (0,1), \quad y \in R,$$

where $a(.) \in L_1(0,1)$ and $b(.)$ is measurable and bounded.

Now, for the local existence of the solutions we have the following theorem:

Theorem 3.1

If assumptions (i) - (iii) are satisfied, such that

$$\frac{\sup | \ b(t) \ |}{\Gamma(2 \ - \ \alpha)} \ < \ 1, \tag{7}$$

then the fractional order integral equation (4) has a solution $y \in B_r$, where

$$r \ \leq \ \frac{\frac{u_0}{\Gamma(2 \ - \ \alpha)} \ + \ \frac{1}{\Gamma(2 \ - \ \alpha)} \ || \ a \ ||}{1 \ - \ \frac{\sup | \ b(t) \ |}{\Gamma(2 \ - \ \alpha)}}.$$

Proof. Let u be an arbitrary element in B_r. Then from the assumptions (i) - (iii), we have

$$\begin{aligned}
||Ty|| \ &= \ \int_0^1 \ | \ Ty(t) \ | \ dt \\
&\leq \ \int_0^1 \ | \ \frac{u_0}{\Gamma(1 \ - \ \alpha)} \ t^{- \ \alpha} \ | \ dt \ + \ \int_0^1 \ | \ \int_0^t \ \frac{(t \ - \ s)^{- \ \alpha}}{\Gamma(1 \ - \ \alpha)} \ f(s, y(s)) \ ds \ | \ dt \\
&\leq \ \left(\frac{u_0 \ t^{1 \ - \ \alpha}}{\Gamma(2 \ - \ \alpha)} \right)_0^1 \ + \ \int_0^1 \ \int_s^1 \ \frac{(t \ - \ s)^{- \ \alpha}}{\Gamma(1 \ - \ \alpha)} \ dt \ | \ f(s, y(s)) \ | \ ds \\
&\leq \ \frac{u_0}{\Gamma(2 \ - \ \alpha)} \ + \ \int_0^1 \ \frac{(t \ - \ s)^{1 \ - \ \alpha}}{\Gamma(2 \ - \ \alpha)} |_s^1 \ (\ | \ a(s) \ | \ + \ | \ b(s) \ | \ | \ y(s) \ | \ ) \ ds \\
&\leq \ \frac{u_0}{\Gamma(2 \ - \ \alpha)} \ + \ \int_0^1 \ \frac{(1 \ - \ s)^{1 \ - \ \alpha}}{\Gamma(2 \ - \ \alpha)} \ (\ | \ a(s) \ | \ + \ | \ b(s) \ | \ | \ y(s) \ | \ ) \ ds \\
&\leq \ \frac{u_0}{\Gamma(2 \ - \ \alpha)} \ + \ \frac{1}{\Gamma(2 \ - \ \alpha)} \ \int_0^1 \ (\ | \ a(s) \ | \ + \ | \ b(s) \ | \ | \ y(s) \ | \ ) \ ds \\
&\leq \ \frac{u_0}{\Gamma(2 \ - \ \alpha)} \ + \ \frac{1}{\Gamma(2 \ - \ \alpha)} \ || \ a \ || \ + \ \frac{1}{\Gamma(2 \ - \ \alpha)} \ \sup | \ b(t) \ | \ || \ y \ ||.
\end{aligned}$$

therefore the operator T maps L_1 into itself. Now, let $y \in \partial B_r$, that is, $||y|| = r$, then the last inequality implies

$$||Ty|| \ \leq \ \frac{u_0}{\Gamma(2 \ - \ \alpha)} \ + \ \frac{1}{\Gamma(2 \ - \ \alpha)} \ || \ a \ || \ + \ \frac{1}{\Gamma(2 \ - \ \alpha)} \ \sup | \ b(t) \ | \ r.$$

Then $T(\partial B_r) \subset \bar{B}_r$ (closure of B_r) if

$$||Ty|| \ \leq \ \frac{u_0}{\Gamma(2 \ - \ \alpha)} \ + \ \frac{1}{\Gamma(2 \ - \ \alpha)} \ || \ a \ || \ + \ \frac{1}{\Gamma(2 \ - \ \alpha)} \ \sup | \ b(t) \ | \ r \ \leq \ r,$$

which implies that

$$\frac{u_0}{\Gamma(2 - \alpha)} + \frac{1}{\Gamma(2 - \alpha)} \, || \, a \, || + \frac{1}{\Gamma(2 - \alpha)} \, \sup | \, b(t) \, | \, r \leq r.$$

Therefore

$$r \leq \frac{\frac{u_0}{\Gamma(2 - \alpha)} + \frac{1}{\Gamma(2 - \alpha)} \, || \, a \, ||}{1 - \frac{\sup | \, b(t) \, |}{\Gamma(2 - \alpha)}}.$$

From inequality (7) we deduce that $r > 0$. Also, since

$$\begin{aligned} ||f|| &= \int_0^1 | \, f(s, y(s)) \, | \, ds \\ &\leq \int_0^1 (\, | \, a(s) \, | + | \, b(s) \, | \, | \, y(s) \, | \,) \, ds \\ &\leq ||a|| + \sup | \, b(t) \, | \, ||y||. \end{aligned}$$

Then f in $L_1(0,1)$.
Further, from (assumption (i)) f is continuous in y and since I^{α} maps $L_1(0,1)$ continuously into itself, then $I^{\alpha} f(t, y(t))$ is continuous in y. Since y is an arbitrary element in B_r,then T maps B_r into $L_1(0,1)$ continuously.
Now, we will show that T is compact, by using Theorem 2.3. So, let Ω be a bounded subset of B_r. Then $T(\Omega)$ is bounded in $L_1(0,1)$, i.e. condition (i) of Theorem 2.3 is satisfied. It remains to show that $(Ty)_h \to Ty$ in $L_1(0,1)$ when $h \to 0$, uniformly.

$$\begin{aligned} ||(Ty)_h - Ty|| &= \int_0^1 | \, (Ty)_h(t) - (Ty)(t) \, | \, dt \\ &= \int_0^1 | \, \frac{1}{h} \int_t^{t+h} (Ty)(s) \, ds - (Ty)(t) \, | \, dt \\ &\leq \int_0^1 \left(\frac{1}{h} \int_t^{t+h} | \, (Ty)(s) - (Ty)(t) \, | \, ds \right) dt \\ &\leq \int_0^1 \frac{1}{h} \int_t^{t+h} | \, \frac{u_0}{\Gamma(1 - \alpha)} \, s^{- \alpha} - \frac{u_0}{\Gamma(1 - \alpha)} \, t^{- \alpha} \, | \, ds \, dt \\ &+ \int_0^1 \frac{1}{h} \int_t^{t+h} | \, I^{1 - \alpha} \, f(s, y(s)) - I^{1 - \alpha} \, f(t, y(t)) \, | \, ds \, dt. \end{aligned}$$

Since $f \in L_1(0,1)$, then $I^{1 - \alpha} f(.) \in L_1(0,1)$. Moreover, since $t^{- \alpha} \in L_1(0,1)$. Then, we have (see [9])

$$\frac{1}{h} \int_t^{t+h} | \, \frac{u_0}{\Gamma(1 - \alpha)} \, s^{- \alpha} - \frac{u_0}{\Gamma(1 - \alpha)} \, t^{- \alpha} \, | \, ds \to 0$$

and

$$\frac{1}{h} \int_t^{t+h} | \, I^{1 - \alpha} \, f(s, y(s)) - I^{1 - \alpha} \, f(t, y(t)) \, | \, ds \to 0$$

for a.e. $t \in (0,1)$. Therefore, by Theorem 2.3, we have that $T(\Omega)$ is relatively compact, that is, T is a compact operator.
Therefore, Theorem 2.1 with $U = B_r$ and $E = L_1(0,1)$ implies that T has a fixed point. This completes the proof.

Now, for the existence of global solution, we will prove he following theorem :

Theorem 3.2

Let the conditions (i) - (iii) be satisfied in addition to the following condition:

(b) Assume that every solution $y(.) \in L_1(0,1)$ to the equation

$$y(t) = \gamma \left(\frac{u_o}{\Gamma(1 - \alpha)} t^{-\alpha} + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1 - \alpha)} f(s, y(s))\, ds \right) \text{ a.e. on } (0,1),\ 0 < \alpha < 1$$

satisfies $||y|| \neq r$ (r is arbitrary but fixed).

Then the fractional order integral equation (4) has at least one solution $y \in L_1(0,1)$.
Proof. Let y be an arbitrary element in the open set $B_r = \{y : ||y|| < r, r > 0\}$. Then from the assumptions (i) - (iii), we have

$$||Ty|| \leq \frac{u_0}{\Gamma(2 - \alpha)} + \frac{1}{\Gamma(2 - \alpha)} || a || + \frac{1}{\Gamma(2 - \alpha)} \sup | b(t) | \, || y ||.$$

The above inequality means that the operator T maps B_r into L_1. Moreover, we have

$$||f|| \leq ||a|| + \sup | b(t) | \, ||y||.$$

This estimation shows that f in $L_1(0,1)$.
Then from Theorem 3.1 we get that T maps B_r into $L_1(0,1)$ continuously, and the operator T is compact.
Set $U = B_r$ and $D = E = L_1(0,1)$, then from assumption (b), we find that condition $A2$ of Theorem 2.2 does not hold. Therefore, Theorem 2.2 implies that T has a fixed point. This completes the proof.

4 Uniqueness of the solution

Theorem 4.1

If the function $f : (0,1) \times R \to R$ satisfy assumption (ii) of Theorem 3.1 and satisfy the following assumption

$$| f(t,y) - f(t,z) | \leq L \, | y - z |, \tag{8}$$

then the fractional order integral equation (4) has a unique solution.
Proof. From assumption (8), we get

$$| f(t,y) - f(t,0) | \leq L \, | y |,$$

but since

$$| f(t,y) | - | f(t,0) | \le | f(t,y) - f(t,0) | \le L | y |,$$

therefore

$$| f(t,y) | \le | f(t,0) | + L | y |,$$

i.e. assumptions (i) and (iii) of theorem 3.1 are satisfied.
Now, let $y_1(t)$ and $y_2(t)$ be any two solutions of equation (4), then

$$| y_2(t) - y_1(t) | \le L \int_0^t \frac{(t - s)^{-\alpha}}{\Gamma(1 - \alpha)} | y_2(s) - y_1(s) | ds.$$

Therefore

$$\int_0^1 | y_2(t) - y_1(t) | dt \le L \int_0^1 \int_0^t \frac{(t - s)^{-\alpha}}{\Gamma(1 - \alpha)} |y_2(s) - y_1(s)| ds\, dt,$$

$$|| y_2 - y_1 || \le L \int_0^1 \int_s^1 \frac{(t - s)^{-\alpha}}{\Gamma(1 - \alpha)} dt\, |y_2(s) - y_1(s)| ds$$

$$\le \frac{L}{\Gamma(2 - \alpha)} || y_2 - y_1 ||.$$

which implies that

$$y_1(t) = y_2(t).$$

Now for the existence and uniqueness of the solution of problems (1) - (2) and (1) - (3), we have the following two theorems:

Theorem 4.2

If the assumptions of theorem 4.1 are satisfied, then problem (1) - (2) has a unique solution.
Proof. Since

$$u(t) = u_0 + I\, f(t, y(t)) \qquad \text{from (5)},$$

then from conditions (2), we get

$$u_0\, (\eta^\alpha - 1) = \int_0^1 (1 - s)^\alpha f(s, y(s))\, ds - \int_0^\eta (\eta - s)^\alpha f(s, y(s))\, ds,$$

$$u_0 = \int_0^1 G(\eta,\ s)\, f(s, y(s))\, ds,$$

where

$$G(\eta, s) = \begin{cases} \frac{[(1-s)]^\alpha - (\eta-s)^\alpha}{\eta^\alpha - 1} & 0 \le s \le \eta \le 1, \\ \frac{(1-s)]^\alpha}{\eta^\alpha - 1} & 0 \le \eta \le s \le 1. \end{cases}$$

Therefore,

$$u(t) = \int_0^1 G(\eta,\ s)\, f(s, y(s))\, ds + I\, f(t, y(t)),$$

which completes the proof.

Theorem 4.3

If the assumptions of theorem 4.1 are satisfied, then problem (1) - (3) has a solution.
Proof. Since

$$u(t) \;=\; u_0 \;+\; I\; f(t, y(t)) \qquad \text{from (5)},$$

then from conditions (3), we get

$$u_0\; (\eta^{1\;-\;\alpha}\;-\;1) \;=\; \int_0^1 f(s, y(s))\; ds \;-\; \int_0^\eta \eta^{1\;-\;\alpha}\; f(s, y(s))\; ds,$$

$$u_0 \;=\; \int_0^1 G(\eta,\; s)\; f(s, y(s))\; ds,$$

where

$$G(\eta, s) = \begin{cases} -1 & 0 \;\leq\; s \;\leq\; \eta \;\leq\; 1, \\ \frac{1}{\eta^{1\;-\;\alpha}\;-\;1} & 0 \;\leq\; \eta \;\leq\; s \;\leq\; 1. \end{cases}$$

Therefore,

$$u(t) \;=\; \int_0^1 G(\eta,\; s)\; f(s, y(s))\; ds \;+\; I\; f(t, y(t)),$$

which completes the proof.

References

[1] Deimling, K. *Nonlinear Functional Analysis*, Springer-Verlag (1985).

[2] Dugundji, J. Granas, A. *Fixed Point theory*, Monografie Mathematyczne, PWN, Warsaw (1982).

[3] El-Sayed, A. M. A. and Abd El-Salam, Sh. A. On the stability of a fractional-order differential equation with nonlocal initial condition, *EJQTDE*, 29 (2008) 1-8.

[4] El-Sayed, A. M. A. and Abd El-Salam, Sh. A. Nonlocal boundary value problem of a fractional-order functional diffrential equation, *Inter. J. of Non. Sci.* (2009).

[5] Miller, K. S. and Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley, New York (1993).

[6] Podlubny, I. and EL-Sayed, A. M. A. On two definitions of fractional calculus, *Preprint UEF 03-96 (ISBN 80-7099-252-2), Slovak Academy of Science-Institute of Experimental phys.* (1996).

[7] Podlubny, I. *Fractional Differential Equations*, Acad. press, San Diego-New York-London (1999).

[8] Samko, S., Kilbas, A. and Marichev, O. L. *Fractional Integrals and Derivatives*, Gordon and Breach Science Publisher, (1993).

[9] Swartz, C. *Measure, Integration and Function spaces*, World Scientific, Singapore (1994).