DYNAMICS OF A NON-AUTONOMOUS DIFFERENCE EQUATION

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ABSTRACT. In this paper we investigate the boundedness, the periodicity character and the global behavior of the positive solutions of the difference equation

$$x_{n+1} = a_n + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, ...,$$

where $\{a_n\}$ is a sequence of nonnegative real numbers and the initial conditions x_{-1}, x_0 are arbitrary positive real numbers.

1. INTRODUCTION

Difference equations appear as natural descriptions of observed evolution, phenomena because most measurements of time evolving variables are discrete and as such those equations are in their own right important mathematical models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations. This is especially true in the case of Lyapunov theory of stability. Nonetheless, the theory of difference equations is a lot richer than the corresponding theory of differential equations. For example; a simple difference equation resulting from a first order differential may have a phenomena often called appearance of "ghost" solutions or existence of chaotic orbits that can only happen for higher order differential equations and the theory of difference equations is interesting in itself.

Existence of the solutions of difference equations of deferent orders and the study of their qualitative properties such as locally, boundedness, global stability, the periodicity have been discussed by many authors. See for examples [1-8], [10-14] and [17-23].

Our aim in this paper is to discuss the behavior of the positive solutions of the difference equations:

$$x_{n+1} = a_n + \frac{x_n^p}{x_{n-1}^p}, \quad n \geq 0 \ . \tag{1.1}$$

where $\{a_n\}$ is a sequence of positive real numbers and the initial conditions x_{-1}, x_0, and p are arbitrary positive real numbers. In this survey we consider three cases of the sequence a_n.

Key words and phrases. Boundedness character, dynamics, periodic solution, global stability.

Our results generalize and complement some of the previous results in the literatures. Moreover, some examples are given to illustrate the main results.

Here we recall some basic definitions and elementary results that will be usefull in our study of Eq.(1).

Let J be an interval real numbers and let $g : J^{k+1} \times J \to J$, where g is a continuously differentiable function. Consider the difference equation

$$y_{n+1} = g(y_n, y_{n-1}, ..., y_{n-k}), \quad n \geq 0, \tag{1.2}$$

where $y_{-k}, y_{-k+1}, ..., y_0 \in J$. An equilibrium point of Eq.(1.2) is a point $\overline{y} \in R$ such that $\overline{y} = g(\overline{y}, \overline{y}, ..., \overline{y})$. That is, $\overline{y}$ is a fixed point of the function $g(y, y, ..., y)$.

Definition: (Periodicity)

(a) A sequence $\{y_n\}$ is said to be periodic with period p if

$$y_{n+p} = y_n \quad \text{for } n = 0, 1, ... \tag{1.3}$$

(b) A sequence $\{y_n\}$ is said to be periodic with prime period p, or with minimal period p, if it is periodic with period p and p is the least positive integer for which (1.3) holds.

Definition: (Permanence)

A difference equation (1.2) is said to be permanent and bounded if there exists numbers m and M, with $0 < m \leq M < \infty$,such that for any initial conditions $y_{-k}, y_{-k+1}, ..., y_0 \in (0, \infty)$ there exists a positive integer N which depends on the initial conditions such that

$$m < y_n < M, \quad for\ all \quad n \geq N.$$

The linearized equation of Eq.(1.2) about the equilibrium point $\overline{y}$ is

$$z_{n+1} = a_1 z_n + a_2 z_{n-1} + ... + a_{k+1} z_{n-k}, \tag{1.4}$$

where $a_i = \frac{\partial f}{\partial y_{n-i}}(\overline{y}, \overline{y}, ..., \overline{y})$, $i = 0, 1, ..., k$. The characteristic equation of $Eq.$(1.4) is

$$\lambda^{k+1} - \sum_{i=1}^{k+1} a_i \lambda^{k-i+1} = 0.$$

Definition: (Stability)

(i) The equilibrium point $\overline{y}$ of $Eq.$(1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ so for all $y_{-k}, y_{-k+1}, ..., y_0 \in J$ with

$$|y_{-k} - \overline{y}| + |y_{-k+1} - \overline{y}| + ... + |y_0 - \overline{y}| < \delta,$$

we have

$$|y_n - \overline{y}| < \epsilon, \quad \text{for all } n \geq -k.$$

(ii) The equilibrium point $\overline{y}$ of $Eq.$(1.2) is globally asymptotically stable if $\overline{y}$ is locally stable and there exists $\lambda > 0$,such that for all $y_{-k}, y_{-k+1}, ..., y_0 \in J$ with

$$|y_{-k} - \overline{y}| + |y_{-k+1} - \overline{y}| + ... + |y_0 - \overline{y}| < \lambda,$$

we have

$$\lim_{n \to \infty} y_n = \overline{y}.$$

(iii) The equilibrium point $\overline{y}$ of $Eq.(1.2)$ is global attractor if for all $y_{-k}, y_{-k+1}, ..., y_0 \in J$,
we have

$$\lim_{n\to\infty} y_n = \overline{y}.$$

(iv) The equilibrium point $\overline{y}$ of $Eq.(1.2)$ is globally asymptotically stable if $\overline{y}$ is locally stable, and $\overline{y}$ is also a global attractor of $Eq.(1.2)$.

(iiv) The equilibrium point $\overline{y}$ of $Eq.(1.2)$ is unstable if $\overline{y}$ is not locally stable.

Definition: (a) **A positive semicycle** of a solution $\{y_n\}$ of Eq.(1.2) consists of a "string"of terms $\{y_j, y_{j+1}, ..., y_n\}$, all greater than or equal to the equilibrium $\overline{y}$,with $j \geq -1$ and $n \leq \infty$ and such that

$$\text{either } j = -1, \quad \text{or} \quad j > -1 \quad \text{and} \quad y_{j-1} < \overline{y},$$

and

$$\text{either } n = \infty, \quad \text{or} \quad n < \infty \quad \text{and} \quad y_{n+1} < \overline{y}.$$

(b) **A negative semicycle** of a solution $\{y_n\}$ of $Eq.(1.2)$ consists of a "string"of terms $\{y_k, y_{k+1}, ..., y_n\}$, all less than to the equilibrium $\overline{y}$,with $k \geq -1$ and $n \leq \infty$ and such that

$$\text{either } k = -1, \quad \text{or} \quad k > -1 \quad \text{and} \quad y_{k-1} \geq \overline{y},$$

and

$$\text{either } n = \infty, \quad \text{or} \quad n < \infty \quad \text{and} \quad y_{n+1} \geq \overline{y}.$$

Definition: (Oscillatory)

(a) A sequence $\{y_n\}$ is called to oscillate about zero or simply to oscillate if the terms y_n are neither eventually all positive nor eventually all negative. Otherwise, the sequence is called nonoscillatory. A sequence $\{y_n\}$ is called strictly oscillatory if for every $n_0 \geq 0$,there exists $n_1, n_2 \geq n_0$ such that $y_{n1} y_{n2} < 0$.

(b) A sequence $\{y_n\}$ is called to oscillate about $\overline{y}$ if the sequence $\{y_n - \overline{y}\}$ oscillates.

(c) A sequence $\{y_n\}$ is said strictly oscillatory about $\overline{y}$ if the sequence $\{y_n - \overline{y}\}$ is strictly oscillatory.

Let J be some interval real numbers and let $g : J \times J \to J$ be a continuously differentiable function. Then for every set of initial conditions $x_0, x_{-1} \in J$, the difference equation

$$y_{n+1} = g(y_n, y_{n-1}), \qquad n = 0, 1, ... \tag{1.5}$$

has a unique solution $\{y_n\}_{n=-1}^{\infty}$. The linearized equation of Eq.(1.5) is

$$z_{n+1} = a_1 z_n + a_2 z_{n-1}.$$

Theorem A [15] A (linearized stability).

(a) If both roots of the quadratic equation

$$\lambda^2 - a_1\lambda - a_2 = 0, \tag{1.6}$$

lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\overline{y}$ of Eq.(1.5) is locally asymptotically stable.

(b) If at least of the roots of Eq.(1.6) has absolute value greater that one , then the equilibrium $\overline{y}$ of Eq.(1.5) is unstable.

(c) A necessary and sufficient condition for both roots of Eq.(1.6) to lie in the open unit disk $|\lambda| < 1$, is

$$|a_1| < 1 - a_2 < 2.$$

Here the locally asymptotically stable equilibrium $\overline{y}$ is also called a sink.

(d) A necessary and sufficient condition for one root of Eq.(1.6) to have absolute value great than one and for the other to have absolute values less than one is

$$|a_1| > |1 - a_2| \text{ and } a_1^2 + 4a_2 > 1.$$

In this case $\overline{y}$ is called a saddle point.

Theorem B [15] Let $[c, d]$ be an interval of real numbers and assume that

$$f : [c, d] \times [c, d] \to [c, d]$$

is a continuous function satisfying the following properties:

(a) $g(x, y)$ is non-decreasing in $x \in [c, d]$ for each $y \in [c, d]$, and $g(x, y)$ is non-increasing in $y \in [c, d]$ for each $x \in [c, d]$.

(b) If $(m, M) \in [c, d] \times [c, d]$ is a solution of the system

$$f(m, M) = m, \quad \text{and} \quad f(M, m) = M \ ,$$

then $m = M$. Then $Eq.$(1.5) has a unique equilibrium $\overline{y} \in [c, d]$ and every solution of $Eq.$(1.5) converges to $\overline{y}$.

Theorem C [16] Assume that $a_1, a_2, ..., a_{k+1} \in R$. Then

$$\sum_{i=1}^{k+1} | a_i |< 1$$

is a sufficient condition for the locally stability of Eq.(1.2).

Consider the vector difference equation

$$X_{n+1} = H(X_n), \quad n = 0, 1, ... \ . \tag{1.7}$$

where $X_n \in R^{k+1}$ for every $n \geq 0$ and $H \in C^1[R^{k+1}, R^{k+1}]$. Then by translating the equilibrium $\overline{X}$ to $0 \in R^{k+1}$ one can see that linearized equation associated with Eq.(1.7) is given by $Y_{n+1} = AY_N$, where A is the Jacobian matrix $DH(\overline{X})$ of the function H evaluated at the equilibrium $\overline{X}$.

Theorem D [16] Let $\overline{X}$ be an equilibrium point of Eq.(1.7) and assume that H is a C^1 function in R^{k+1}. Then the following statements are true:

(a) If all the eigenvalues of the Jacobian matrix $DH(\overline{X})$ lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{X}$ of Eq.(1.7) is asymptotically stable.

(b) If at least one eigenvalues of the Jacobian matrix $DH(\overline{X})$ has absolute value greater that one, then the equilibrium $\overline{X}$ of Eq.(1.7) is unstable.

Theorem E [16] Consider the difference equation

$$x_{n+1} = f(x_n, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.8}$$

where $f \in C[(0,\infty)^{k+1},(0,\infty)]$ is increasing in each of its arguments, where the initial conditions $x_{-k},...,x_0$ are positive. Assume that Eq.(1.8) has a unique positive equilibrium $\overline{x}$, and suppose that the function h defined by

$$h(x) = f(x,x,...,x), \quad y \in (0,\infty),$$

satisfies

$$(h(x)-x)(x-\overline{x}) < 0, \text{ for } x \neq \overline{x}.$$

Then $\overline{x}$ is a global attractor of all positive solutions of Eq.(1.8).

Theorem F [9] Let J be some interval of real numbers, $f \in C[J^{v+1}, J]$, and let $\{x_n\}_{n=-v}^{\infty}$ be a bounded solution of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-v}), \quad n = 0,1,..., \tag{1.9}$$

with

$$I = \lim_{n\to\infty} \inf x_n, \quad S = \lim_{n\to\infty} \sup x_n, \text{ with } I, S \in J.$$

Let Z denote the set of all integers $\{..., -1, 0, 1, ...\}$. Then there exist two solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-v}), \tag{1.10}$$

which satisfy the equation for all $n \in Z$, with

$$I_0 = I, \ S_0 = S, \quad \text{and} \quad I_n, S_n \in [I,S], \quad \text{for all } n \in Z,$$

and such that for every $N \in Z$, I_N and S_N are limit points of $\{x_n\}_{n=-v}^{\infty}$. Therefore, for every $m \leq -v$ there exist two subsequences $\{x_{r_n}\}$ and $\{x_{l_n}\}$ of the solution $\{x_n\}_{n=-v}^{\infty}$ so the following are true:

$$\lim_{n\to\infty} x_{r_n+N} = I_N, \quad \text{and} \quad \lim_{n\to\infty} x_{l_n+N} = S_N, \ N \geq m.$$

The solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of Eq.(1.10) are called **Full limiting solutions** of Eq.(1.10) associated with the solution $\{x_n\}_{n=-v}^{\infty}$ of Eq.(1.9).

This paper is divided into two parts. **Part I** deals with the Eq.(1.1) when $p = 1$. **Part II** concerned with Eq.(1.1) when p is a positive real number.

Part I: Studing of Eq.(1.1) with P=1

Here our goal is to consider the local stability, the boundedness character, and the global asymptotic behavior of the positive solutions of the difference equation:

$$x_{n+1} = a_n + \frac{x_n}{x_{n-1}}, \quad n \geq 0, \tag{1.11}$$

where $\{a_n\}$ is a sequence of nonnegative real numbers and the initial condition x_{-1}, andx_0 are positive real numbers.

In the following we consider three cases of the sequence $\{a_n\}$.

2. Case 1. When $\lim_{n\to\infty} a_n = a$

Permanence of Eq.(1.11)

In this subsection we investigate the boundedness of Eq.(1.11).

Theorem 1. *Suppose that* $\lim_{n\to\infty} a_n = a \geq 1$, *at that point every positive solution of Eq.(1.11) is bounded and persists.*

Proof. Suppose that $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(1.11). Then

$$x_n \geq a > 1, \;\; for\ all \;\; n \geq 1.$$

Let $\epsilon \in (0, a-1)$, we see from Eq.(1.11) that

$$x_n \geq a - \epsilon, \quad \text{for all } n \geq -1.$$

Then we can find $L \in (a+\epsilon, a+\epsilon+1)$ such that

$$L - \epsilon \leq x_{-1}, x_0 \leq \frac{L-\epsilon}{L-a-\epsilon}.$$

Since $a > 1$, then we get

$$a \leq \frac{L-\epsilon-1}{L-\epsilon-a}.$$

Set

$$f(u,v) = a + \frac{u}{v}.$$

Then

$$f(L-\epsilon, \frac{L-\epsilon}{L-a-\epsilon}) = a + \frac{L-\epsilon}{\frac{L-\epsilon}{L-a-\epsilon}} = L - \epsilon,$$

and

$$f(\frac{L-\epsilon}{L-a-\epsilon}, L-\epsilon) = a + \frac{\frac{L-a-\epsilon}{L-\epsilon}}{L-\epsilon} \leq a + \frac{1}{L-a-\epsilon} \leq \frac{L-\epsilon}{L-a-\epsilon}.$$

Now it follows from Eq.(1.11) that

$$x_1 = f(x_0, x_{-1}) \leq f(\frac{L-\epsilon}{L-a-\epsilon}, L-\epsilon) \leq \frac{L-\epsilon}{L-a-\epsilon}.$$

Again we see from Eq.(1.11) that

$$x_1 = f(x_0, x_{-1}) \geq f(L-\epsilon, \frac{L-\epsilon}{L-a-\epsilon}) = L - \epsilon.$$

By induction we obtain that

$$L - \epsilon \leq x_n \leq \frac{L-\epsilon}{L-a-\epsilon}, \quad \text{for all} \;\; n = -1, 0, 1, \ldots.$$

Second assume that $a = 1$ and let $\epsilon \in (0, \delta)$ and $\delta \in (0,1)$, it follows from Eq.(1.11) that

$$x_n \geq 1 - \epsilon + \delta \;, \quad for \;\; n \geq 1.$$

Then one can find $L \in (1+\epsilon+\delta, 2+\epsilon+\delta)$ such that

$$L - \epsilon + \delta \leq x_{-1}, x_0 \leq \frac{L-\epsilon+\delta}{L-\epsilon-1+\delta}.$$

In this way whatever is left of the proof is like the above and it is overlooked. □

Global Attractity of Eq.(1.11)

Here, we show that if $a > 1$, Therefore every positive solution of Eq.(1.11) converges to $(a+1)$.

Theorem 2. *Assume that $a \geq 1$. At that point each positive solution of Eq.(1.11) converges to the unique positive equilibrium point $\overline{x} = (a+1)$ of Eq.(1.11).*

Proof. Note, when $a \geq 1$, it was shown in **Theorem 1.2.1** that each positive solution of Eq.(1.11) is bounded. Then we have the following

$$s = \lim_{n\to\infty} \inf x_n, \quad \text{and} \quad S = \lim_{n\to\infty} \sup x_n.$$

It is clear that $s \leq S$. We want to proof that $s \geq S$. Now it is easy to see from Eq.(1.11) that

$$s \geq a + \frac{s}{S}, \quad \textit{and} \quad S \leq a + \frac{S}{s}.$$

Thus we have

$$sS \geq aS + s, \quad \text{and} \quad sS \leq as + S.$$

This implies

$$aS + s \leq as + S.$$

Then we get

$$a(S-s) \leq (S-s),$$

or

$$(a-1)(S-s) \leq 0 \Leftrightarrow s \geq S.$$

Thus the proof is complete. □

Example 1. *Figure (1) shows the global attractivity of the equilibrium point $\overline{x} = 2$ of Eq.(1.11) whenever $x_{-1} = 1.21, x_0 = 1.32,$ and $a = 1$.*

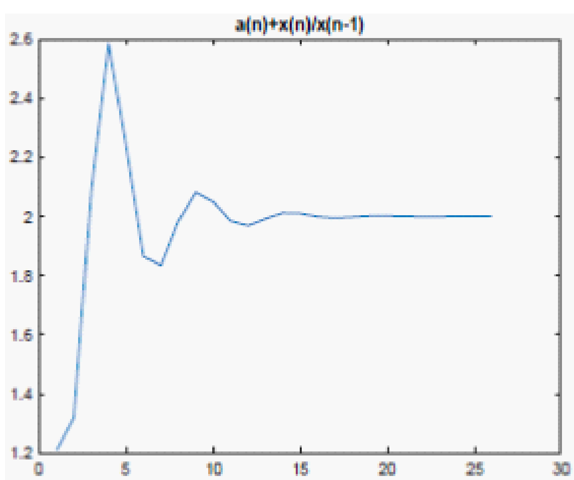

Figure (1)

Example 2. *Figure (2) shows the global attractivity of the equilibrium point $\overline{x} = 6$ of Eq.(1.11) whenever $x_{-1} = 4$, $x_0 = 9$, and $a = 5$.*

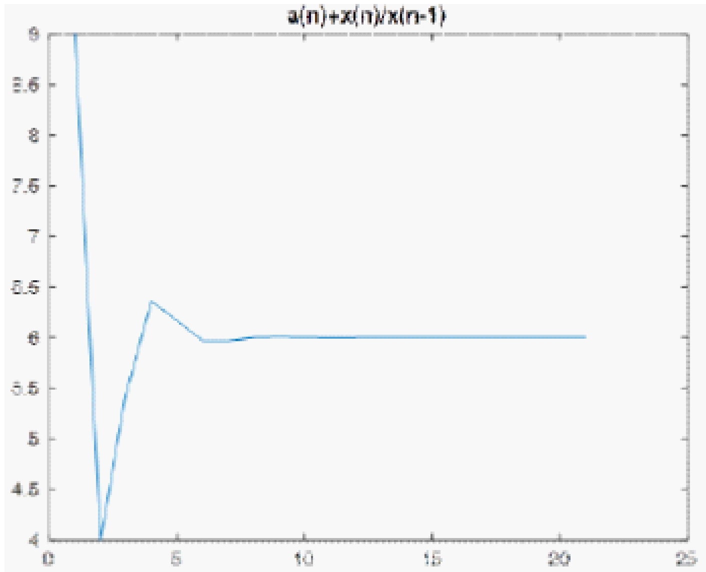

Figure (2)

3. Case 2. When a_n is periodic

In this subsection we research the periodicity character of the positive solutions of Eq.(1.11) whenever $\{a_n\}$ is a periodic sequence of period two of the form $\{\alpha, \beta, \alpha, \beta, ...\}, \alpha \neq \beta$. Assume that $a_{2n} = \alpha$, and $a_{2n+1} = \beta$. Then we have

$$x_{2n+1} = \alpha + \frac{x_{2n}}{x_{2n-1}}, \tag{3.1}$$

and

$$x_{2n+2} = \beta + \frac{x_{2n+1}}{x_{2n}}, \quad n \geq 0. \tag{3.2}$$

Periodicity of the solutions

Here we investigate the periodic solutions of Eq.(1.11).

Theorem 3. *Assume that $\{a_n\} = \{\alpha, \beta, \alpha, \beta, ...\}$, with $\alpha \neq \beta$. Then Eq.(1.11) has periodic solution of prime period two.*

Proof. Let $\{x_n\}$ be a solution of Eq.(1.11), with the initial values x_{-1}, and x_0 such that

$$x_{-1} = \frac{\alpha x_{-1} + x_0}{x_{-1}}, \quad \textit{and} \quad x_0 = \frac{\beta x_0 + x_{-1}}{x_0}. \tag{3.3}$$

Let $x_{-1} = x$, and $x_0 = y$, then we obtain from (3.3)

$$x = \alpha + \frac{y}{x}, \quad \text{and} \quad y = \beta + \frac{x}{y}. \tag{3.4}$$

Now we want to prove that (3.4) has a solution (x, y), $x > 0$, $y > 0$. From the first relation of (3.4) we get

$$y = (x - \alpha)x. \tag{3.5}$$

From (3.5) and the second relation of (3.4) we obtain

$$x(x-\alpha) = \beta + \frac{x}{x(x-\alpha)},$$

or

$$x(x-\alpha)^2 - \beta(x-\alpha) - 1 = 0.$$

Now define the function

$$f(x) = x(x-\alpha)^2 - \beta(x-\alpha) - 1, \quad x > \alpha. \tag{3.6}$$

Then

$$\lim_{x\to\alpha^+} f(x) = -1, \quad \textit{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

Hence Eq.(3.6) has at least one solution $x > \alpha$. Then if $y = (x-\alpha)x$, we have that the solution $\{x_n\}_{n=-1}^{\infty}$ is periodic of prime period two. ☐

Example 3. *Figure (3) shows that the solution of Eq.(1.11) is periodic solution of period two when* $x_{-1} = 1.34$, $x_0 = 3.210$, $\alpha = 1$, *and* $\beta = 0.1$.

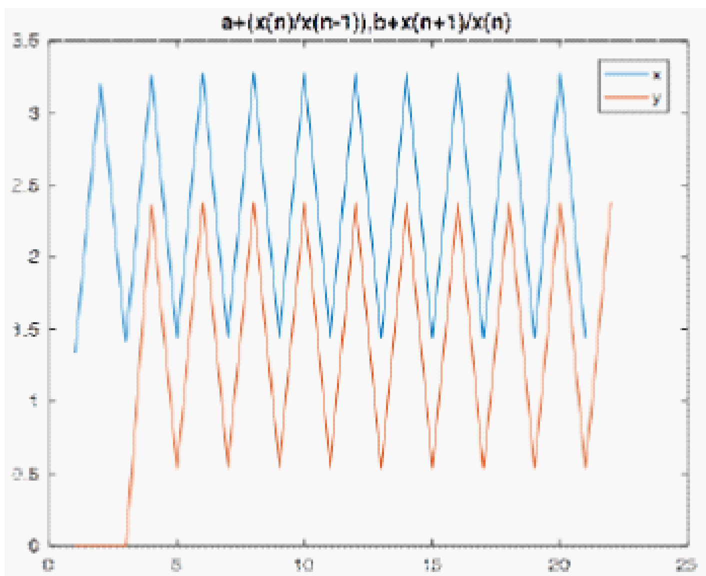

Figure (3)

Local Stability of the periodic solutions

Here we investigate the local stability character of Eq.(1.11).

Theorem 4. *Assume that* $\{x_n\}_{n=-1}^{\infty}$ *be a periodic solution of period two of Eq.(1.11) and consider Eq.(1.11) when the case* $\{a_n\} = \{\alpha, \beta, \alpha, \beta, ...\}$ *with* $\alpha \neq \beta$. *Suppose that*

$$\frac{\alpha}{\beta^2} + \frac{1}{\alpha\beta} + \frac{1}{\alpha^3} < \frac{\alpha}{x}.$$

Then $\{x_n\}_{n=-1}^{\infty}$ *is locally asymptotically stable.*

Proof. It was shown in **Theorem 1.3.1** that there exist x, y such that

$$x = \alpha + \frac{y}{x}, \quad \text{and} \quad y = \beta + \frac{x}{y}. \tag{3.7}$$

Now Eq.(1.11) can be rewritten in the following form by splitting the even-indexed and odd-indexed terms:

$$\begin{aligned} u_{n+1} &= \alpha + \tfrac{v_n}{u_n}, \\ v_{n+1} &= \beta + \tfrac{\alpha u_n + v_n}{u_n v_n}. \end{aligned} \tag{3.8}$$

Now, we consider the map T on $[0,\infty)\times[0,\infty)$ such that

$$T(u,v) = \begin{bmatrix} T_1(u,v) \\ T_2(u,v) \end{bmatrix} = \begin{bmatrix} \alpha + \frac{v}{u} \\ \beta + \frac{\alpha u + v}{uv} \end{bmatrix}.$$

Then we have

$$\frac{\partial T_1}{\partial u} = \frac{-v}{u^2}, \quad \text{and} \quad \frac{\partial T_1}{\partial v} = \frac{1}{u},$$
$$\frac{\partial T_2}{\partial u} = \frac{-v^2}{v^2 u^2}, \quad \text{and} \quad \frac{\partial T_2}{\partial v} = \frac{-\alpha u^2}{u^2 v^2},$$

Therefore the Jacobian matrix of T at (x,y) is

$$J_T(x,y) = \begin{bmatrix} \frac{-y}{x^2} & \frac{1}{x^2} \\ \frac{-1}{x^2} & \frac{-\alpha}{y^2} \end{bmatrix},$$

and its characteristic equation associated with (x,y) is

$$\lambda^2 + \lambda\left(\frac{\alpha}{y^2} + \frac{y}{x^2}\right) + \frac{\alpha}{x^2 y} + \frac{1}{x^3} = 0. \tag{3.9}$$

It follows from (3.7) that $\frac{y}{x^2} = 1 - \frac{\alpha}{x}$ and since $x > \alpha$, and $y > \beta$ we have

$$\frac{\alpha}{y^2} + \frac{y}{x^2} + \frac{\alpha}{x^2 y} + \frac{1}{x^3} < \frac{\alpha}{\beta^2} + \frac{1}{\alpha\beta} + \frac{1}{\alpha^3} + 1 - \frac{\alpha}{x} < 1.$$

Thus

$$\frac{\alpha}{\beta^2} + \frac{1}{\alpha\beta} + \frac{1}{\alpha^3} < \frac{\alpha}{x} < 1.$$

Then all roots of Eq.(3.9) have modulus less than 1. Therefore by **Theorem D** that System (3.8) is asymptotically stable. The proof is complete. □

Theorem 5. *Assume that $\{a_n\} = \{\alpha,\beta,\alpha,\beta,...\}$, with $\alpha \neq \beta$. Then every solution of Eq.(1.11) converges to a period two solution of Eq.(1.11).*

Proof. We know by **Theorem 1.2.1** that every positive solution of Eq.(1.11) is bounded, it follows that there are some positive constants l, L, s, and S such that

$$l = \lim_{n\to\infty} \inf x_{2n+1}, \quad \text{and} \quad L = \lim_{n\to\infty} \sup x_{2n+1},$$
$$s = \lim_{n\to\infty} \inf x_{2n}, \quad \text{and} \quad S = \lim_{n\to\infty} \sup x_{2n}.$$

Then it is easy to see from Eq.(3.1) and Eq.(3.2) that

$$l \geq \alpha + \frac{s}{L}, \quad \text{and} \quad L \leq \alpha + \frac{S}{l},$$

and

$$s \geq \beta + \frac{l}{S}, \quad \text{and} \quad S \leq \beta + \frac{L}{s}.$$

Then we obtain

$$Ll \geq \alpha L + s, \quad \text{and} \quad Ll \leq \alpha l + S,$$

and

$$Ss \geq \beta S + l, \quad \text{and} \quad Ss \leq \beta s + L.$$

Thus we get

$$\alpha L + s \le Ll \le \alpha l + S, \quad \text{and} \quad \beta S + l \ \le Ss \le \beta s + L.$$

Thus we have

$$\alpha(L - l) \le S - s, \quad \text{and} \quad \beta(S - s) \le L - l. \tag{3.10}$$

Thus it is clear from (3.10) that $s = S$ and $l = L$. Now suppose $\lim_{n\to\infty} x_{2n+1} = S$, and $\lim_{n\to\infty} x_{2n} = L$. We want to proof that $S \neq L$. From Eq.(3.1) and Eq.(3.2) we get

$$S = \alpha + \frac{L}{S}, \quad \text{and} \quad L = \beta + \frac{S}{L}.$$

As that sake of contradiction assume that $L = S$, then

$$L = \alpha + 1, \quad \text{and} \quad S = \beta + 1$$

thus $\alpha = \beta$ which is a contradiction. So $\lim_{n\to\infty} x_{2n+1} \neq \lim_{n\to\infty} x_{2n}$. The proof is so complete. □

Example 4. *Figure (4) shows the global attractivity of the equilibrium point of Eq.(1.11) is when* $x_{-1} = 2.3$, $x_0 = 1.3$, $\alpha = 0.73827543$, *and* $\beta = 0.6763772$.

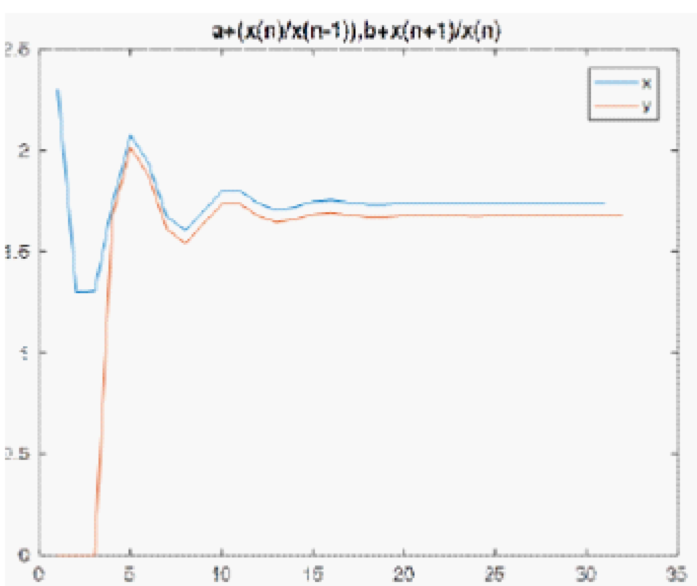

Figure (4)

Example 5. *Figure (5) shows every solution of Eq.(1.11) converges to a period two solution when* $x_{-1} = 15.30$*,* $x_0 = 10.30$*,* $\alpha = 6$*, and* $\beta = 1$.

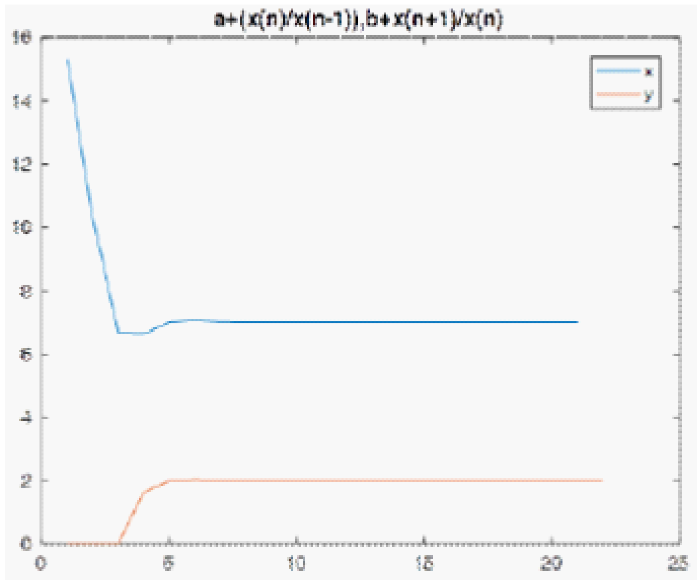

Figure (5)

4. Case 3. The autonomous case of Eq.(1.11)

Consider Eq.(1.11) with $a_n = a$, where $a \in (0, \infty)$ then Eq.(1.11) has the form

$$x_{n+1} = a + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, ..., \tag{4.1}$$

where the initial conditions x_{-1}, x_0 are arbitrary positive numbers. Clearly, the only equilibrium point of Eq.(4.1) is $\overline{x} = a + 1$.

The linearized equation of Eq.(4.1) about the equilibrium point $\overline{x} = a + 1$ is

$$y_{n+1} - \frac{1}{a+1} y_n + \frac{1}{a+1} y_{n-1} = 0.$$

Local Stability

In this subsection we deal the local stability of Eq.(4.1).

Lemma 1. *The following statements are true.*

1. The equilibrium point $\overline{x} = a + 1$ *of Eq.(4.1) is locally asymptotically stable if* $a > 1$.

2. The equilibrium point $\overline{x} = a + 1$ *of Eq.(4.1) is unstable if* $0 \leq a \leq 1$.

Proof. The proof is followed directly by **Theorem A** and so will be omitted. □

4.1. **Boundedness.** Here, we investigate the bounded character of Eq.(4.1).

Theorem 6. *Suppose that $a > 1$, then every positive solution of Eq.(4.1) is bounded.*

Proof. It follows from Eq.(4.1) that

$$x_{2n+1} = a + \tfrac{x_{2n}}{x_{2n-1}},$$
$$x_{2n} = a + \tfrac{x_{2n-1}}{x_{2n-2}}.$$

Therefore

$$x_{2n-1} > a, \quad \text{and} \quad x_{2n-2} > a, \quad \text{for every } n \geq 1.$$

Then

$$x_{2n+1} = a + \frac{x_{2n}}{x_{2n-1}} < a + \frac{x_{2n}}{a}, \quad \text{and} \quad x_{2n} = a + \frac{x_{2n-1}}{x_{2n-2}} < a + \frac{x_{2n-1}}{a}$$

Then it follows by induction that

$$x_{2n+1} < a + (1 + \frac{1}{a} + \frac{1}{a^2} + ...) + \frac{x_{-1}}{a^n} = a + \frac{a}{a-1} + \frac{x_{-1}}{a^n},$$

and

$$x_{2n} < a + (1 + \frac{1}{a} + \frac{1}{a^2} + ...) + \frac{x_0}{a^n} = a + \frac{a}{a-1} + \frac{x_0}{a^n}.$$

The result now follows. □

Theorem 7. *Assume that $a > 1$ then every solution of Eq.(4.1) is bounded and persists.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(4.1), then

$$x_{n+1} = a + \frac{x_n}{x_{n-1}} > a, \quad \text{for all } n \geq 1. \tag{4.2}$$

Again it follows from Eq.(4.1) that

$$x_{n+1} = a + \frac{x_n}{x_{n-1}} \leq a + \frac{x_n}{a}.$$

Then

$$\limsup x_n \leq \frac{a}{1 - \frac{1}{a}} = \frac{a^2}{a-1}. \tag{4.3}$$

Then the result follows from (4.2) and (4.3). □

Global attractor

In the following Theorem, we establish sufficient conditions for global attractor of Eq.(4.1).

Theorem 8. *Assume that $a > 1$.Then the equilibrium point $\overline{x} = a + 1$ is a global attractor of Eq.(4.1).*

Proof. Let $f : [c,d]^2 \to [c,d]$ be a function defined by $f(u,v) = a + \frac{u}{v}$. Assume that (m, M) is a solution of the system

$$m = f(m, M), \quad \text{and} \quad M = f(M, m).$$

Then we get

$$(a-1)(M-m) = 0,$$

Since $a > 1$, then we obtain

$$M = m.$$

It follows by **Theorem B** that $\overline{x}$ is a global attractor of Eq.(4.1) and then the proof is complete. □

Remark 1. *In case 3 this case has been treated by many others such as [Amleh]. Here we give an alternative proofs of our results.*

Part II : Studying of Eq.(1.1)

In this part we investigate the behavior of the positive solutions of the difference equation

$$x_{n+1} = a_n + \frac{x_n^p}{x_{n-1}^p}, \qquad \text{for } n \geq 0,$$

where p is a positive real number, a_n is a positive sequence and the initial conditions x_{-1}, x_0 are positive real numbers.

5. Case 1. When $a_n = a \in R^+$

In this case Eq.(1.1) takes the form

$$x_{n+1} = a + \frac{x_n^p}{x_{n-1}^p}, \quad n = 0, 1, 2... . \tag{5.1}$$

Local Stability of the Equilibrium Points

At the present we discuss the local stability character of the solutions of Eq.(5.1).

It is easy to see that the only positive equilibrium point of Eq.(5.1) is given by $\overline{x} = a + 1$. Let $f : (0, \infty)^2 \to (0, \infty)$ be a function defined by

$$f(x, y) = a + \frac{x^p}{y^p}.$$

Therefore

$$\frac{\partial f(x,y)}{\partial x} = \frac{p x^{p-1}}{y^p}, \qquad \text{and} \qquad \frac{\partial f(x,y)}{\partial y} = -\frac{p x^p}{y^{p+1}}.$$

We see that

$$\frac{\partial f(\overline{x},\overline{x})}{\partial x} = \frac{p}{a+1} = p_1, \qquad \text{and} \qquad \frac{\partial f(\overline{x},\overline{x})}{\partial y} = -\frac{p}{a+1} = p_2 \ .$$

Then the linearized equation of Eq.(5.1) about $\overline{x}$ is

$$y_{n+1} - \frac{p}{a+1} y_n + \frac{p}{a+1} y_{n-1} = 0.$$

Theorem 9. *The following statements are valid:*

(i) if $p < a+1$, furthermore the positive equilibrium point $\overline{x}$ of Eq.(5.1) is locally asymptotically stable, and is called a sink.

(ii) If $p > a+1$, then the positive equilibrium point $\overline{x}$ of Eq.(5.1) is unstable, and is called a repeller.

(iii) If $p = a+1$, then the positive equilibrium point $\overline{x}$ of Eq.(5.1) is unstable, and is called a nonhyperbolic point.

Proof. (i) We set $p_1 = \frac{p}{\overline{x}}$, and $p_2 = -\frac{p}{\overline{x}}$. So by **Theorem A (a)**

$$\mid p_1 \mid -1 + p_2 < 0 \Leftrightarrow \frac{p}{a+1} - \frac{p}{a+1} - 1 < 0 \Longleftrightarrow -1 < 0.$$

Also

$$1 + p_2 - 2 < 0 \Longleftrightarrow -1 + \frac{p}{a+1} < 0 \Leftrightarrow \frac{p}{a+1} < 1.$$

which is valid iff

$$p < a+1.$$

So $\overline{x}$ is locally asymptotically stable when $p < a+1$.

(ii) By **Theorem A (d)** we have

$$\mid p_2 \mid -1 = \frac{p}{a+1} - 1 > 0 \Longleftrightarrow \frac{p}{a+1} > 1,$$

and

$$\mid p_1 \mid - \mid 1 - p_2 \mid = \frac{p}{a+1} - 1 - \frac{p}{a+1} = -1.$$

Thus $\overline{x}$ is unstable (repeller point) when $p > a+1$.

(iii) By **Theorem A (e)** we have

$$p_2 = -1 \Leftrightarrow -\frac{p}{1+a} = -1 \Leftrightarrow -p = -(a+1) \Leftrightarrow p = a+1,$$

and

$$\mid p_1 \mid -2 \leq 0 \Leftrightarrow \frac{p}{a+1} - 2 \leq 0 \Leftrightarrow p \leq 2(a+1).$$

Thus $\overline{x}$ is unstable (repeller point) when $p > a+1$. □

Boundedness of Solutions of Eq.(5.1)

In this subsection we discuss the suffiction conditions for bounded solution of Eq.(5.1).

Theorem 10. *If $0 < p < 1$, consequently every positive solution of Eq.(5.1) is bounded and persists.*

Proof. We obtain from Eq.(5.1) that

$$x_{n+1} > a, \quad n \geq 0.$$

Hence $\{x_n\}$ persists. It follows again of Eq.(5.1) that

$$x_{2n+1} \leq a + (\frac{x_{2n}}{a})^p, \quad n = 0, 1, \dots .$$

Now we suppose the difference equation

$$y_{n+1} = a + (\frac{y_n}{a})^p, \quad n \geq 0. \tag{5.2}$$

Let $\{y_n\}$ be a solution of Eq.(5.2) with $y_0 = x_0$. Thus, clearly

$$x_{2n+1} \leq y_{n+1} \quad (resp \quad x_{2n+2} \leq y_{n+1}), \qquad n = 0, 1, \dots .$$

We will establish that the sequence $\{y_n\}$ is bounded. Let

$$f(x) = a + \frac{x^p}{a^p}.$$

Then

$$f^{'}(x) = \frac{1}{a^p} p x^{p-1} > 0, \quad \text{and} \quad f^{''}(x) = \frac{1}{a^p} p(p-1) x^{p-2} < 0.$$

Therefore the function f is increasing and concave . Thus we obtain that there is a unique fixed point y^* of the equation $f(y) = y$. Likewise the function f satisfies

$$(f(y) - y)(y - y^*) < 0, \qquad y \in (0, \infty).$$

By **Theorem E** y^* is a global attractor of all positive solutions of Eq.(5.2) and so $\{y_n\}$ is bounded. Then from Eq.(5.2) the sequence $\{x_n\}$ is so bounded. This finishes the proof of the theorem. □

Example 6. *Figure (6) shows the bounded solutions of the equilibrium point* $\overline{x} = 24$ *of Eq.(5.1) whenever* $x_{-1} = 1.0323$, $x_0 = 2.441, a = 23,$ *and* $p = 0.000000002$.

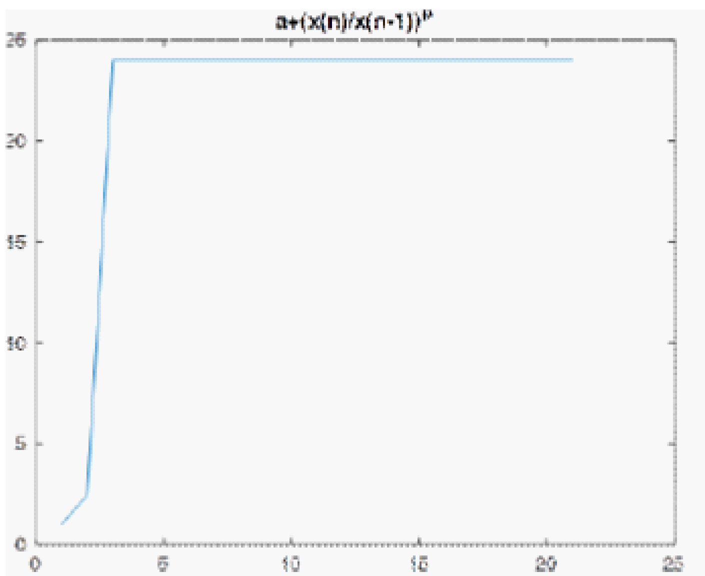

Figure (6)

Theorem 11. *Assume that* $p \geq 4$. *Then Eq.(5.1) has unbounded solutions.*

Proof. Note that for every solution $\{x_n\}_{n=-1}^{\infty}$of Eq.(5.1) the following inequality holds:

$$x_{n+1} > \frac{x_n^p}{x_{n-1}^p}, \text{ for } n \in N. \tag{5.3}$$

Let $y_n = \ln x_n$. It follows from (5.3) that

$$y_{n+1} - p y_n + p y_{n-1} > 0. \tag{5.4}$$

Note that the roots of the polynomial

$$p(\lambda) = \lambda^2 - p\lambda + p,$$

are given by

$$\lambda_1, \lambda_2 = \frac{p \pm \sqrt{p^2 - 4p}}{2},$$

Since $p \geq 4$ we have that $\lambda_1 > 1$. On the other hand we have

$$\lambda_2 = \frac{2p}{p + \sqrt{p^2 - 4p}}.$$

Hence if $p \geq 4$, both roots of $p(\lambda)$ are positive. Note that (5.4) can be rewritten in the form

$$y_{n+1} - \lambda_1 y_n - \lambda_2(y_n - \lambda_1 y_{n-1}) > 0.$$

Then we see that

$$\frac{x_{n+1}}{x_n^{\lambda_1}} > (\frac{x_n}{x_{n-1}^{\lambda_1}})^{\lambda_2}. \tag{5.5}$$

It follows that

$$\frac{x_n}{x_{n-1}^{\lambda_1}} > (\frac{x_{n-1}}{x_{n-2}^{\lambda_1}})^{\lambda_2} > ... > (\frac{x_1}{x_0^{\lambda_1}})^{\lambda_2} > (\frac{x_0}{x_{-1}^{\lambda_1}})^{\lambda_2}.$$

Select x_{-1} and x_0 so that

$$x_0 > 1, \quad x_0 = x_{-1}^{\lambda_1}.$$

Then it follows by (5.5) that

$$x_n > (\frac{x_0}{x_{-1}^{\lambda_1}})^{\lambda_2} x_{n-1}^{\lambda_1} = x_{n-1}^{\lambda_1} > ... > x_0^{\lambda_1^n},$$

and consequently $x_n > x_0^{\lambda_1^n}$, $n \in N$. Letting $n \to \infty$, then $x_n \to \infty$. From which the outcome takes after. $\square$

Global Stability of Eq.(5.1)

Here we study the characteristic task of global stability of Eq.(5.1).

Theorem 12. *Suppose that $a \geq 1$ and $0 < p < 1$. Then the unique positive equilibrium point of Eq.(5.1) is globally asymptotically stable.*

Proof. By **Theorem 1.5.1 (i)** $\overline{x}$ is locally asymptotically stable. Thus it is suffices prove that every positive solution of Eq.(5.1) tends to the unique positive equilibrium $\overline{x}$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(5.1). By **Theorem 1.5.2** $\{x_n\}_{n=-1}^{\infty}$ is bounded. Thus we have

$$a \leq s = \liminf x_n, \quad \text{and} \quad S = \limsup x_n < \infty .$$

Then we get from (5.1)

$$S \leq a + \frac{S^p}{s^p}, \quad \text{and} \quad s \geq a + \frac{s^p}{S^p}. \tag{5.6}$$

We claim that $S = s$, otherwise $S > s$. We obtain from (5.6)

$$s^p S \leq s^p a + S^p, \quad \text{and} \quad sS^p \geq S^p a + s^p. \tag{5.7}$$

Thus we have

$$s^{1-p} < S^{1-p},$$

or equivalently

$$sS^p < Ss^p. \tag{5.8}$$

It follows from Eq.(5.7) and Eq.(5.8) that

$$S^p a + s^p \leq s^p a + S^p.$$

Hence

$$S^p(a-1) \leq s^p(a-1).$$

which is impossible for $a \geq 1$. Hence the result follows. □

Example 7. *Figure (7) shows the global attractivity of the equilibrium point* $\overline{x} = 1.2000$ *of Eq.(5.1) whenever* $x_{-1} = 1.03$, $x_0 = 2.441$, $a = 1.1$, *and* $p = 0.9$.

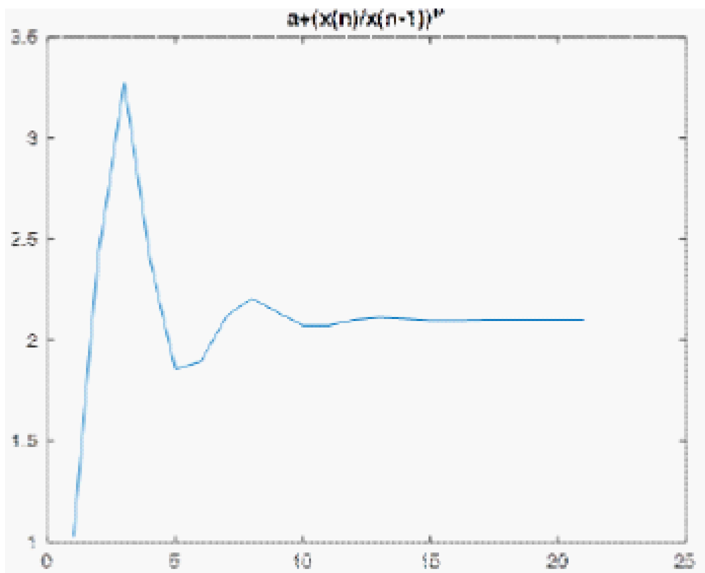

Figure (7)

Oscillatory Solutions of Eq.(5.1)

Here we present the characteristic task of oscillatory solution of Eq.(5.1).

Theorem 13. *Assume that* $0 < p \leq 1$, *then every positive solution of Eq.(5.1) oscillates about the equilibrium point* $\overline{x} = a + 1$ *with semicycles of length two or three and the extreme of every semicycle occurs at the first or the second term.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(5.1). First, we present every positive semicycle except possibly the first term has two or three terms. Assume that $x_{N-1} < \overline{x}$, and $x_N \geq \overline{x}$, for some $N \in \mathbb{N}$. We obtain

$$x_{N+1} = a + \frac{x_N^p}{x_{N-1}^p} > a + 1 = \overline{x}.$$

If $x_{N+1} > x_N$, so we get

$$x_{N+2} = a + \frac{x_{N+1}^p}{x_N^p} > a + 1 = \overline{x}.$$

Since $p \in (0,1]$, we include that

$$x_{N+2} = a + \frac{x_{N+1}^p}{x_N^p} \leq a + \frac{x_{N+1}^p}{\overline{x}^p} \leq a + \frac{x_{N+1}^p}{a+1} \leq x_{N+1}.$$

So $\overline{x} < x_{N+2} < x_{N+1}$. Therefore

$$x_{N+3} = a + \frac{x_{N+2}^p}{x_{N+1}^p} < a + 1 = \overline{x}.$$

Then the proof is completed. □

Theorem 14. *Eq.(5.1) has no periodic solutions of prime period two.*

Proof. As the sake of contradiction. Assume that $..., x, y, x, y, ...$ be a periodic solution of period two of Eq.(5.1). It press that

$$x = a + (\frac{y}{x})^p, \quad \text{and} \quad y = a + (\frac{x}{y})^p, \tag{5.9}$$

which suggest that

$$y = a + \frac{1}{x-a}. \tag{5.10}$$

Substituting from (5.9) into (5.10) and after some calculation we get

$$(x-a)^{p+1}x^p = (a(x-a)+1)^p. \tag{5.11}$$

Taking the logarithm on both sides of (5.11), we acquire

$$f(x) = (p+1)\ln(x-a) + p\ln x - p\ln[a(x-a)+1] = 0. \tag{5.12}$$

It is obvious that $x = a+1$ is an obvious solution of (5.12). Presently we examine that this is the unique solution of the equation (5.12). Now

$$\acute{f}(x) = \frac{(x-a)(ax + p(a(x-a)+1)) + (p+1)x}{x(x-a)(a(x-a)+1)}.$$

Thus $\acute{f}(x) > 0$, for $x \in (a, \infty)$, which implies that the f is strictly increasing on the interval (a, ∞). Hence, the equilibrium point $\overline{x} = a+1$ is the unique solution of (5.12). From Eq.(5.11) we obtain $y = a+1$ and consequently. This means $(a+1, a+1)$ is the unique solution of System (5.9). Finishing the proof of the theorem. □

6. Case 2. When a_n be a periodic sequence of period two

In this section we study the behavior of solution of Eq.(1.1) while a_n is a periodic sequence of period two with $\alpha, \beta \in (0, \infty)$ and $\alpha \neq \beta$. Consider $a_{2n} = \alpha$, and $a_{2n+1} = \beta$. Then we have

$$\begin{aligned} x_{2n+1} &= \alpha + \tfrac{x_{2n}^p}{x_{2n-1}^p}, \\ x_{2n+2} &= \beta + \tfrac{x_{2n+1}^p}{x_{2n}^p}. \end{aligned} \tag{6.1}$$

Now Eq.(1.1) can be rewritten in the following form:

$$\begin{aligned} u_{n+1} &= \alpha + \tfrac{u_n^p}{v_n^p}, \\ v_{n+1} &= \beta + \tfrac{v_n^p}{u_n^p}. \end{aligned} \tag{6.2}$$

Locally stability

Here we discuss the local stability of System (6.2). It is easy to see that $(\overline{u}, \overline{v}) = (\alpha+1, \beta+1)$ is the unique positive equilibrium point of System (6.2).

Theorem 15. *If $p < \frac{(\beta+1)(\alpha+1)}{(\alpha+1)^p(\beta+1)^p}$, then the positive equilibrium point $(\overline{u}, \overline{v}) = (\alpha+1, \beta+1)$ of System (6.2) is locally asymptotically stable.*

Proof. We consider the map T on $[0, \infty) \times [0, \infty)$ such that

$$T(u, v) = \begin{bmatrix} T_1(u, v) \\ T_2(u, v) \end{bmatrix} = \begin{bmatrix} \alpha + \frac{u^p}{v^p} \\ \beta + \frac{v^p}{u^p} \end{bmatrix}.$$

Then we have

$$\frac{\partial T_1(u, v)}{\partial u} = -\frac{pu^{p-1}v^p}{(u^p)^2}, \quad \text{and} \quad \frac{\partial T_1(u, v)}{\partial v} = \frac{pv^{p-1}}{u^p},$$

and

$$\frac{\partial T_2(u, v)}{\partial u} = \frac{pu^{p-1}}{v^p}, \quad \text{and} \quad \frac{\partial T_2(u, v)}{\partial v} = -\frac{pv^{p-1}u^p}{(v^p)^2}.$$

Therefore the Jacobian matrix of T at $(\overline{u}, \overline{v}) = (\alpha+1, \beta+1)$ is

$$J(E_{\alpha,\beta}) = \begin{bmatrix} -\frac{pu^{p-1}v^p}{(u^p)^2} & \frac{pv^{p-1}}{u^p} \\ \frac{pu^{p-1}}{v^p} & -\frac{pv^{p-1}u^p}{(v^p)^2} \end{bmatrix},$$

and the characteristic equation associated with $(\overline{u}, \overline{v})$ is

$$p(\lambda) = \lambda^2 - \lambda p(\frac{(\beta+1)^{p-1}}{(\alpha+1)} + \frac{(\alpha+1)^{p-1}}{(\beta+1)}).$$

Then we obtain

$$\lambda_1 = 0, \quad \lambda_2 = p(\frac{(\beta+1)^{p-1}}{(\alpha+1)} + \frac{(\alpha+1)^{p-1}}{(\beta+1)}).$$

It follows by **Theorem D** that the equilibrium point $(\overline{u}, \overline{v}) = (\alpha+1, \beta+1)$ of System (6.2) is locally asymptotically stable if $p < \frac{(\beta+1)(\alpha+1)}{(\alpha+1)^p+(\beta+1)^p}$. Then the proof is completed. □

Example 8. *Figure (8) shows the local stability of the equilibrium point*

$(\overline{u}, \overline{v}) = (21.6073, 0.0780)$ *of System (6.2) whenever* $u_0 = 2.43, v_0 = 0.4562$, $\alpha = 0.76$, $\beta = 0.03$, *and* $p = 0.54$.

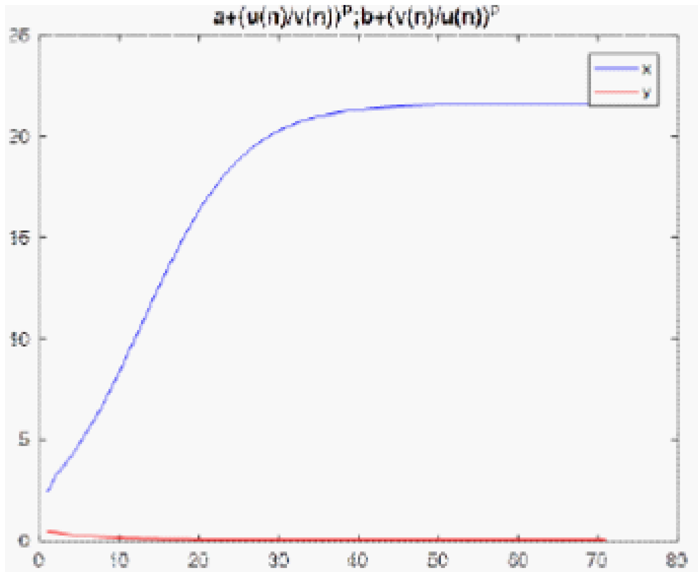

Figure (8)

Periodicity of Eq.(1.1)

In this subsection we investigate the excitons of periodic solutions of Eq.(1.1).

Theorem 16. *Assume that* $\{a_n\} = \{\alpha, \beta, \alpha, \beta, ...\}$, *with* $\alpha \neq \beta$. *Then Eq.(1.1) has periodic solution of prime period two.*

Proof. To prove that Eq.(1.1) possess a periodic solution $\{x_n\}$ of prime period two, we must find positive numbers x_{-1}, x_0 such that

$$x_{-1} = \frac{\alpha x_{-1}^p + x_0^p}{x_{-1}^p}, \quad \textit{and} \quad x_0 = \frac{\beta x_0^p + x_{-1}^p}{x_0^p}. \tag{6.3}$$

Let $x_{-1} = x$, and $x_0 = y$, then we obtain from (6.3)

$$x = \alpha + \frac{y^p}{x^p}, \quad \text{and} \quad y = \beta + \frac{x^p}{y^p}. \tag{6.4}$$

Now we want to prove that (6.4) has a solution (x, y), $x > 0$, $y > 0$. From the first relation of (6.4) we have

$$y = (x - \alpha)^{\frac{1}{p}} x. \tag{6.5}$$

From (6.5) and the second relation of (6.4) we get

$$x(x - \alpha)^{\frac{1}{p}} = \beta + \frac{x^p}{x^p(x - \alpha)},$$

or

$$x(x - \alpha)^{\frac{p+1}{p}} - \beta(x - \alpha) - 1 = 0.$$

Now define the function

$$f(x) = x(x - \alpha)^{\frac{p+1}{p}} - \beta(x - \alpha) - 1, \quad x > \alpha. \tag{6.6}$$

Then

$$\lim_{x\to\alpha^+} f(x) = -1, \quad \textit{and} \quad \lim_{x\to\infty} f(x) = \infty.$$

Hence Eq.(6.6) has at least one solution $x > \alpha$. Then if $y = (x-\alpha)^{\frac{1}{p}}x$, we have that the solution $\{x_n\}_{n=-1}^{\infty}$ is periodic of prime period two. □

7. Case 3. When a_n is a positive bounded sequence

In this section we assume that a_n is a positive bounded sequence

$$\lim_{n\to\infty} \inf a_n = a \geq 0, \qquad \text{and} \qquad \lim_{n\to\infty} \sup a_n = b < \infty. \tag{7.1}$$

Boundedness

The primary theorem indicate to the boundedness and the persistence of the positive solutions of Eq.(1.1).

Theorem 17. *Assume* $0 < p < 1$. *Therefore every positive solution of Eq.(1.1) is bounded and persists.*

Proof. The proof is similar to the proof of **Theorem 1.5.2** and will be omitted. □

Lemma 2. *Assume that* $0 < p \leq 1$. *Let* $\lim_{n\to\infty} \inf a_n = a \geq 0$, *and* $\lim_{n\to\infty} \sup a_n = b < \infty$ *and* $\{x_n\}$ *be a positive solution of Eq.(1.1). Then*

$$\frac{ab-1}{b-1} \leq \lim_{n\to\infty} \inf x_n \leq \lim_{n\to\infty} \sup x_n \leq \frac{ab-1}{a-1}.$$

Proof. Assume

$$\lim_{n\to\infty} \inf x_n = \lambda, \qquad \text{and} \qquad \lim_{n\to\infty} \sup x_n = \mu. \tag{7.2}$$

Let $\epsilon > 0$ for $n \geq N_0(\epsilon)$ we get

$$\lambda - \epsilon \leq x_n \leq \mu + \epsilon, \qquad \text{and} \qquad a - \epsilon \leq a_n \leq b + \epsilon.$$

Therefore

$$x_{n+1} \geq a - \epsilon + (\frac{\lambda-\epsilon}{\eta+\epsilon})^p. \tag{7.3}$$

Taking the $\lim_{n\to\infty}$ inf for Eq.(7.3). We obtain

$$\lambda \geq a - \epsilon + (\frac{\lambda-\epsilon}{\eta+\epsilon})^p.$$

Since $\epsilon > 0$ is arbitrary,

$$\lambda \geq a + (\frac{\lambda}{\eta})^p. \tag{7.4}$$

Similarly

$$\eta \leq b + (\frac{\eta}{\lambda})^p. \tag{7.5}$$

We get from equations (7.4) and (7.5) that

$$\lambda\eta^p \geq a\eta^p + \lambda^p, \qquad \text{and} \qquad \eta\lambda^p \leq b\lambda^p + \eta^p. \tag{7.6}$$

Since $0 < p < 1$ holds. Then we have

$$\lambda^{1-p} \leq \eta^{1-p},$$

or equivalently

$$\lambda\eta^p \leq \eta\lambda^p. \tag{7.7}$$

It follows from equations (7.6) and (7.7) that

$$a\eta^p + \lambda^p \leq b\lambda^p + \eta^p.$$

So

$$\eta^p(a-1) \leq \lambda^p(b-1),$$

and we have

$$(\frac{\eta}{\lambda})^p \leq \frac{b-1}{a-1}, \qquad \text{and} \qquad (\frac{\lambda}{\eta})^p \geq \frac{a-1}{b-1}.$$

We obtain from Eq.(7.4) for all $n > N_0(\epsilon)$ that

$$\lambda \geq a + (\frac{\lambda}{\eta})^p \geq a + \frac{a-1}{b-1} = \frac{ab-1}{b-1}.$$

Similarly from Eq.(7.5) we get

$$\eta \leq \frac{ab-1}{a-1}.$$

Thus the proof is completed. □

Now define the sequence $\{y_n\}$ to be

$$y_n = \frac{x_n}{\overline{x}_n}, \; n = -1, 0, 1, ...,$$

where $\overline{x}_n$ be a fixed solution of Eq.(1.1). Then Eq.(1.1) will be rewritten as

$$y_{n+1} = \frac{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p (\frac{y_n}{y_{n-1}})^p}{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p}. \tag{7.8}$$

Lemma 3. *Let $\{\overline{x}_n\}$ be a fixed positive solution of Eq.(7.8). Then the following statements are true.*

(i) Eq.(7.8) has a positive equilibrium solution $\overline{y} = 1$.
(ii) Let $\{y_n\}$ be a solution of Eq.(7.8). Then except possibly for the first semicycle, every solution of Eq.(7.8) has semicycle of length one.

Proof. (i) trivial.
(ii) Assume that for some n, $y_{n-1} \geq y_n$. Then $(\frac{y_n}{y_{n-1}}) < 1$ and

$$y_{n+1} = \frac{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p (\frac{y_n}{y_{n-1}})^p}{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p} < \frac{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p}{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p} = 1. \tag{7.9}$$

Let $\{y_n\}$ be an finally oscillatory solution of Eq.(7.8) such as $y_{n-1} < 1$ and $y_n \geq 1$. From part (7.9) it follows that $y_{n+1} < 1$. Therefore the positive semicycle has exactly one term. The proof for negative semicycle is similar. □

Lemma 4. *Let $\{y_n\}$ be a fixed positive solution of Eq.(7.8). Suppose that there exists an $m \in \{1, 2, ...\}$ such that*

$$y_{2m-1} < 1, \quad \textit{and} \quad y_{2m} \geq 1. \tag{7.10}$$

Then

$$y_{2n-1} < 1, \quad \textit{and} \quad y_{2n} \geq 1, \; \textit{for } n = m, m+1, ... \,. \tag{7.11}$$

Moreover, if

$$y_{2m-1} \geq 1, \quad \textit{and} \quad y_{2m} < 1. \tag{7.12}$$

Then

$$y_{2n-1} \geq 1, \quad \textit{and} \quad y_n < 1, \ \textit{for } n = m, m+1, \ldots \ . \tag{7.13}$$

Proof. Let $\{y_n\}$ be a solution of Eq.(7.8) such that Eq.(7.10) holds for an $m \in \{1, 2, \ldots\}$. we have

$$y_{2m-1} = \frac{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p (\frac{y_n}{y_{n-1}})^p}{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p} \geq \frac{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p}{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p} = 1.$$

Working inductively we can easily prove that Eq.(7.11) is satisfied. Similarly we can prove that if Eq.(7.12) holds for an $m \in \{1, 2, \ldots\}$, then Eq.(7.13) is satisfied. This completes the proof of the lemma. □

Global attractor of the solutions

Here we investigate the global stability of Eq.(1.1).

Theorem 18. *Let $\{\overline{x}_n\}$ be a fixed solution of Eq.(1.1). Suppose that one of the following holds:*

(i) $0 < p \leq \frac{1}{2}$.
(ii) $\frac{1}{2} < p < 1$, $a > 1$, *and* $a(a-1) > b - 1$. *Then for every solution $\{x_n\}$ of Eq.(1.1) the relation $\lim_{n\to\infty} \frac{x_n}{\overline{x}_n} = 1$ is true.*

Proof. (i) Let $\{y_n\}$ be a solution of Eq.(7.8). It is sufficient to prove that

$$\lim_{n\to\infty} y_n = 1$$

Suppose that there exists an $m \in \{1, 2, \ldots\}$ such that (7.10) or (7.12) . Without loss of generality we may assume that (7.10) holds for an $m \in \{1, 2, \ldots\}$ and $0 < p \leq \frac{1}{2}$ is satisfied.

Let

$$\mu = \lim_{n\to\infty} \inf y_n, \quad \text{and} \quad \zeta = \lim_{n\to\infty} \sup y_n. \tag{7.14}$$

also

$$\tau = \lim_{n\to\infty} \inf \overline{x}_n, \quad \text{and} \quad \omega = \lim_{n\to\infty} \sup \overline{x}_n, \tag{7.15}$$

and

$$\delta = \frac{\omega}{\tau}. \tag{7.16}$$

Define the function F by

$$F(x, y, z) = \frac{x + y^p z^p}{x + y^p} \tag{7.17}$$

for $x, y.z > 0$. Then we have

$$\frac{\partial F}{\partial x} = \frac{y^p(1 - z^p)}{(x + y^p)^2}, \quad \text{and} \quad \frac{\partial F}{\partial y} = \frac{pxy^{p-1}(z^p - 1)}{(x + y^p)^2}$$

Let $n \geq m$. Using Eq.(7.8) we have

$$
\begin{aligned}
y_{2n+1} &= F(a_{2n}, \tfrac{\overline{x}_{2n}}{\overline{x}_{2n-1}}, \tfrac{y_{2n}}{y_{2n-1}}),\\
y_{2n+2} &= F(a_{2n+1}, \tfrac{\overline{x}_{2n+1}}{\overline{x}_{2n}}, \tfrac{y_{2n+1}}{y_{2n}}).
\end{aligned}
\tag{7.18}
$$

Since (7.10) holds by Lemma 3 we obtain the following:

$$\frac{y_{2n-1}}{y_{2n}} < 1, \quad \text{and} \quad \frac{y_{2n}}{y_{2n-1}} \geq 1, \quad \text{for } n \geq m.$$

using Eq.(7.1), (7.14)-(7.18) and monotonic properties of F we have

$$\theta \leq F(a, \delta, \frac{\zeta}{\mu}) = \frac{a + (\frac{\zeta}{\mu})^p \delta^p}{a + \delta^p}, \quad \text{and} \quad \mu \geq F(a, \delta, \frac{\mu}{\zeta}) = \frac{a + (\frac{\mu}{\zeta})^p \delta^p}{a + \delta^p},$$

or

$$\zeta \mu^p \leq \frac{a\mu^p + \zeta^p \delta^p}{a + \delta^p}, \quad \text{and} \quad \mu \zeta^p \geq \frac{a\zeta^p + \mu^p \delta^p}{a + \delta^p}.$$

Then

$$a\zeta^p \mu^{p-1} + \mu^{2p-1}\delta^p \leq \zeta^p \mu^p \leq a\mu^p \zeta^{p-1} + \zeta^{2p-1}\delta^p.$$

Hence

$$a\zeta^p \mu^{p-1} + \mu^{2p-1}\delta^p \leq a\mu^p \zeta^{p-1} + \zeta^{2p-1}\delta^p,$$

and so

$$\zeta^p(a\mu^{p-1} + \mu^{p-1}(\frac{\mu}{\zeta})^p \delta^p) \leq \mu^p(a\zeta^{p-1} + \zeta^{p-1}(\frac{\zeta}{\mu})^p \delta^p),$$

or

$$(\frac{\zeta}{\mu})^p(a(\frac{\mu}{\zeta})^{p-1} - \delta^p) \leq a - (\frac{\mu}{\zeta})^{p-1}\delta^p.$$

Thus

$$a\frac{\zeta}{\mu} - \delta^p(\frac{\zeta}{\mu})^p \leq a - (\frac{\zeta}{\mu})^{1-p}\delta^p.$$

Since $0 < p \leq \frac{1}{2}$, we obtain

$$a(\frac{\zeta}{\mu} - 1) \leq \delta^p((\frac{\zeta}{\mu})^p - (\frac{\zeta}{\mu})^{1-p}).$$

Therefore

$$a(\frac{\zeta}{\mu} - 1) \leq 0,$$

which implies that

$$\zeta \leq \mu.$$

Thus we get that $\zeta = \mu$. The proof is completed.

(ii) Now suppose $\frac{1}{2} < p < 1$, $a > 1$, and $a(a-1) > b - 1$. Note that $(\frac{\eta}{\lambda})^p \leq \frac{b-1}{a-1}$ and $(\frac{\lambda}{\eta})^p \geq \frac{a-1}{b-1}$. Then it follows by that (7.1), (7.14-7.18) and $(\frac{\eta}{\lambda})^p \leq \frac{b-1}{a-1}$ and $(\frac{\lambda}{\eta})^p \geq \frac{a-1}{b-1}$ hold. Then we obtain

$$\theta \leq F(a, \frac{\eta}{\lambda}, \frac{\zeta}{\mu}) = \frac{a + (\frac{\eta}{\lambda})^p (\frac{\zeta}{\mu})^p}{a + (\frac{\eta}{\lambda})^p} \leq \frac{a + (\frac{b-1}{a-1})(\frac{\zeta}{\mu})^p}{a + (\frac{b-1}{a-1})},$$

and

$$\mu \geq F(a, \frac{\eta}{\lambda}, \frac{\mu}{\zeta}) = \frac{a + (\frac{\eta}{\lambda})^p (\frac{\mu}{\zeta})^p}{a + (\frac{\eta}{\lambda})^p} \geq \frac{a + (\frac{b-1}{a-1})(\frac{\mu}{\zeta})^p}{a + (\frac{b-1}{a-1})}.$$

Then

$$\mu^p\zeta \leq \frac{a\mu^p}{a+(\frac{b-1}{a-1})}+\frac{(\frac{b-1}{a-1})\zeta^p}{a+(\frac{b-1}{a-1})}, \quad \text{and} \quad \mu\zeta^p \geq \frac{a\zeta^p}{a+(\frac{b-1}{a-1})}+\frac{(\frac{b-1}{a-1})\mu^p}{a+(\frac{b-1}{a-1})}. \tag{7.19}$$

Since $\mu \leq \zeta$ it follows that $\mu\zeta^p \leq \zeta\mu^p$. Therefore from (7.19) we get

$$\frac{a\zeta^p}{a+(\frac{b-1}{a-1})}+\frac{(\frac{b-1}{a-1})\mu^p}{a+(\frac{b-1}{a-1})} \leq \frac{a\mu^p}{a+(\frac{b-1}{a-1})}+\frac{(\frac{b-1}{a-1})\zeta^p}{a+(\frac{b-1}{a-1})}.$$

Then

$$\left(\frac{a}{a+(\frac{b-1}{a-1})}-\frac{(\frac{b-1}{a-1})}{a+(\frac{b-1}{a-1})}\right)\zeta^p \leq \left(\frac{a}{a+(\frac{b-1}{a-1})}-\frac{(\frac{b-1}{a-1})}{a+(\frac{b-1}{a-1})}\right)\mu^p. \tag{7.20}$$

Since $\frac{1}{2} < p < 1$, $a > 1$, and $a(a-1) > b-1$, we obtain from (7.20) that $\zeta \leq \mu$ and so $\zeta = \mu$. Then the proof is completed. □

Periodicity of Eq.(1.1)

In the following theorem we find the sufficient conditions for the existence of two-periodic solutions for Eq.(1.1).

Theorem 19. *Assume that $0 < p < 1$ and $\{a_n\}$ is a periodic sequence of period twos. Then Eq.(1.1) has a periodic solution of prime period two.*

Proof. For Eq.(1.1) posses a periodic solution $\{x_n\}$ of prime period two, we must find some positive numbers x_{-1}, x_0. Assume that $\{a_n\} = \{a_0, a_1, a_0, a_1, ...\}$, such that

$$x_{-1} = x_1 = a_0 + \left(\frac{x_0}{x_{-1}}\right)^p, \text{ and } x_0 = x_2 = a_1 + \left(\frac{x_1}{x_0}\right)^p, \tag{7.21}$$

We shall show that System (7.21) is consistent. We get from Eq.(7.21)

$$(x_{-1} - a_0)(x_0 - a_1) = 1. \tag{7.22}$$

It follows that

$$(x_{-1} - a_0)^{p+1} = \frac{(a_1(x_{-1} - a_0) + 1)^p}{x_{-1}^p}, \text{ and } (x_0 - a_1)^{p+1} = \frac{(a_0(x_0 - a_1) + 1)^p}{x_0^p}. \tag{7.23}$$

We define a function F by

$$F(x) = (x - a_0)^{p+1} - \frac{(a_1(x - a_0) + 1)^p}{x^p}, \quad x > a_0.$$

Then

$$F(a_0) = -\frac{1}{a_0} < 0, \quad \text{and } F(a_0 + 1) = 1 - \frac{(a_1 + 1)^p}{(a_0 + 1)^p} > 0.$$

Now let $a_1 < a_0$, then F has a zero, say x_{-1}, in the interval $(a_0, a_0 + 1)$, and in view of equations (7.22) and (7.23) we get that Eq.(1.1) has a two-periodic solution. Assume now that $a_1 > a_0$. We define a function G such that

$$G(x) = (x - a_1)^{p+1} - \frac{(a_0(x - a_1) + 1)^p}{x^p}, \quad x > a_1.$$

Then

$$G(a_1) = -\frac{1}{a_1} < 0, \quad F(a_1+1) = 1 - \frac{(a_0+1)^p}{(a_1+1)^p} > 0.$$

Thus, G has a zero, say x_0, in the interval (a_1, a_1+1), and in view of equations (7.22) and (7.23) we get that Eq.(1.1) has a two-periodic solution. □

Theorem 20. *Assume that* $\{a_n\} = \{\alpha, \beta, \alpha, \beta, ...\}$, *with* $\alpha \neq \beta$. *Then every solution of Eq.(1.1) converges to a period two solution of Eq.(1.1).*

Proof. We know by **Theorem 1.7.1** that every positive solution of Eq.(1.1) is bounded, therefore there are some positive constants l, L, s and S such that

$$l = \lim_{n\to\infty} \inf x_{2n+1}, \quad \text{and} \quad L = \lim_{n\to\infty} \sup x_{2n+1},$$

$$s = \lim_{n\to\infty} \inf x_{2n}, \quad \text{and} \quad S = \lim_{n\to\infty} \sup x_{2n}.$$

Now we get from Eq.(1.1) that

$$\begin{aligned} x_{2n+1} &= a_{2n} + \tfrac{x_{2n}^p}{x_{2n-1}^p}, \\ x_{2n+2} &= a_{2n+1} + \tfrac{x_{2n+1}^p}{x_{2n}^p}. \end{aligned} \tag{7.24}$$

Therefore, it is easy to see from System (7.24) that

$$l \geq a_0 + \frac{s^p}{L^p}, \quad \text{and} \quad L \leq a_0 + \frac{S^p}{l^p},$$

and

$$s \geq a_1 + \frac{l^p}{S^p}, \quad \text{and} \quad S \leq a_1 + \frac{L^p}{s^p}.$$

Then we obtain

$$L^p l \geq a_0 L^p + s^p, \quad \text{and} \quad L l^p \leq a_0 l^p + S^p,$$

and

$$S^p s \geq a_1 S^p + l^p \ , \quad \text{and} \quad S s^p \leq a_1 s^p + L^p.$$

So, we get

$$a_0 L^p + s^p \leq L l^p \leq L^p l \leq a_0 l^p + S^p,$$

and

$$a_1 S^p + l^p \leq S^p s \leq S s^p \leq a_1 s^p + L^p.$$

Thus, we have

$$a_0(L^p - l^p) \leq S^p - s^p, \quad \text{and} \quad a_1(S^p - s^p) \leq L^p - l^p. \tag{7.25}$$

Thus it is clear from (7.25) that $s = S$ and $l = L$. Now assume that $\lim_{n\to\infty} x_{2n+1} = S$ and $\lim_{n\to\infty} x_{2n} = L$. We want to proof that $S \neq L$. From System(7.24) we get

$$S = \alpha + \frac{L^P}{S^P}, \quad \text{and} \quad L = \beta + \frac{S^P}{L^P}.$$

As the sake of contradiction assume that $L = S$, then

$$L = \alpha + 1, \quad \text{and} \quad S = \beta + 1$$

thus $\alpha = \beta$ which is a contradiction. So $\lim_{n\to\infty} x_{2n+1} \neq \lim_{n\to\infty} x_{2n}$. The proof is so completed. □

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