Discrete spline Numerov method for solving Swift-Hohenberg equation

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Abstract

Discrete spline function based method is developed to solve the time fractional Swift-Hohenberg equation in the sense of Riemann Liouville derivative. Via Fourier method, the developed method is unconditionally stable. Two schemes are acquired, these schemes are verified to be convergent of order two and four. Numerical results are demonstrated for various values of fractional Brownian α as a function of time and also the standard motion $\alpha = 1$ to confirm the applicability and the theoretical results.

Keywords: Discrete spline; Numerov method, Riemann-Liouville fractional derivative; Grünwald-Letnikov derivative; stability analysis, convergence analysis.

1. Introduction

The generality of ordinary differentiation and integration to an arbitrary non-integer order is fractional calculus. For many years, fractional derivatives were not used in physics despite they have a long mathematical history. In the last ten years, fractional calculus starts to attract much more attention of physicists and mathematicians because modern applications recently dominate it in differential and integral equations, physics, fluid mechanics, mathematical biology, electrochemistry, signal processing, oil industries and many other applications [4-5,10,15,17]. Exact solutions of most fractional differential equations cannot be established, so numerical techniques are obligatory to find approximate solutions for these fractional differential equations. Approximate and numerical methods have been set such as variational iteration method, Homotopy perturbation method, Adomian decomposition method, Homotopy analysis method and collocation method [1-3,6-14,25]. This paper is assigned to new and recent application of fractional calculus in science and engineering that is variable time fractional Swift-Hohenberg equation of the form: [16,21-22]

$$D_t^{\alpha(t)}u(x,t) + D_x^4 u(x,t) + 2D_x^2 u(x,t) + (1-\mu)u(x,t) + f(u) = 0, \quad 0 < \alpha(t) \le 1, \qquad (1)$$

with initial conditions and boundary conditions:

$$u(x,0)=0.1\sin\bar{x},\ u(x,t)=D_x^2u(x,t)=0 \ \text{ for } x=0 \text{ and } x=l,\ t>0, \tag{2}$$

where $D_t^{\alpha(t)}u(x,t)$ represents the fractional derivative in the sense of Riemann-Liouville, $D_x^k u(x,t), k=2,4$, refers to the standard derivative of integer order and $\bar{x}=\pi x/l$. Also, the nonlinear term $f(u)$ is assumed locally Lipschitz continuous, that is for some $\theta>0$,

$$\|f(u_1)-f(u_2)\|\le\theta\|u_1-u_2\|. \tag{3}$$

Definition 1 [6,18-20] The Riemann-Liouville fractional derivative of $u(x,t)$ is:

$$D_t^{\alpha(t)}u(x,t)=\frac{1}{\Gamma(k-\alpha(t))}\frac{\partial^k}{\partial t^k}\int_0^t (x-t)^{k-\alpha(t)-1}u(x,z)dz,\ k-1<\alpha(t)<k, \tag{4}$$

where $\Gamma(.)$ is the Gamma function.

Definition 2 [18,21,23-24] The Grünwald-Letnikov fractional derivative of $u(x,t)$ is:

$${}^{GL}D_t^{\alpha(t)}u(x,t)=\lim_{h_t\to 0}\frac{1}{h_t}\sum_{k=0}^{n=[t/h_t]}(-1)^k\binom{\alpha(t)}{k}u(x,t-kh_t)\,, \tag{5}$$

where $[t/h_t]$ is the integer part of t/h_t.

Lemma 1[18] The Grünwald-Letnikov fractional derivative satisfies that:

$$D_t^{\alpha(t)}u(x,t)=h_t^{-\alpha(t)}\Delta_{h_t}^{\alpha(t)}u(x,t)+O(h_t),\qquad h_t\to 0. \tag{6}$$

Then the Grünwald-Letnikov fractional derivative is equivalent to Riemann-Liouville fractional derivative (5) as:

$$D_t^{\alpha(t)}u(x,t)={}^{GL}D_t^{\alpha(t)}u(x,t)\approx h_t^{-\alpha(t)}\Delta_{h_t}^{\alpha(t)}u(x,t) \tag{7}$$

This paper is prearranged as follows: Section 2 is devoted to deriving a discrete cubic spline method. In section 3, stability analysis of our approach is discussed. In section 4 we study the convergence analysis of the proposed scheme. Numerical results are offered to demonstrate the applicability and the accuracy in section 5. Finally, in section 6 we conclude the results of the proposed method.

2 Discrete cubic spline solution

We wish for solving the variable time fractional Swift-Hohenberg equation (1-2). we first define the discrete spline function, let $\omega: 0=x_0<x_1<x_2...<x_n=1$ be a uniform mesh of $[0,1]$ with $\omega=x_i-x_{i-1}, i=1,2,3,...,n$. Following [26], then the discrete cubic spline has the following form:

$$s_i(x,t_j,\omega,h)=\frac{(x_i-x)^{\{3\}}}{6\omega}M_{i-1}^j+\frac{(x-x_{i-1})^{\{3\}}}{6\omega}M_i^j$$

$$+[s_{i-1}^j-\frac{(\omega^2-h^2)}{6}M_{i-1}^j]\frac{(x_i-x)}{\omega}+[s_i^j-\frac{(\omega^2-h^2)}{6}M_i^j]\frac{(x-x_{i-1})}{\omega}, \tag{8}$$

Applying the continuity conditions, we have

$$s_{i-1}^j-2s_i^j+s_{i+1}^j=\gamma M_{i-1}^j+\beta M_i^j+\gamma M_{i+1}^j,\ i=1,2,...,n-1, \tag{9}$$

where

$$\gamma=\frac{\omega^2-h^2}{6}\ \text{ and }\ \beta=\frac{2\omega^2+h^2}{6}\ . \tag{10}$$

For $h=\omega/\sqrt{2}$, the discrete spline Numerov method can be achieved by the scheme (9) as:

$$s_{i-1}^j-2s_i^j+s_{i+1}^j=\frac{\omega^2}{12}[M_{i-1}^j+12M_i^j+M_{i+1}^j],\ i=1,2,...,n-1, \tag{11}$$

Also, for $h\to 0$, scheme (9) reduces to the following scheme of ordinary cubic spline:

$$s_{i-1}^j-2s_i^j+s_{i+1}^j=\frac{\omega^2}{6}[M_{i-1}^j+4M_i^j+M_{i+1}^j],\ i=1,2,...,n-1.$$

Lemma 2[26]

Let s be a cubic spline interpolation of u defined by $s(x_i,t_j)=u_i^j$, and assume that:

$$D_x^m u\in C(0,l)\times(0,T),\ 0\le m\le 6\quad\text{and}\quad D_x^{m^*}uu\in C(0,l)\times(0,T),\ 0\le m^*\le 2,$$

then the next relation holds:

$$\omega^2 u_i^{j(4)}=(M_{i-1}^j-2M_i^j+M_{i+1}^j)+O(\omega^4). \tag{12}$$

Using Eqns (11-12), we acquire that

$$(2\gamma+\beta)\omega^2M_i^j=u_{i-1}^j-2u_i^j+u_{i+1}^j-\gamma\omega^4u_i^{j(4)}+O(\omega^6). \tag{13}$$

Using Eq.(13) in Eq.(11) , we get:

$$u_{i-2}^j-4u_{i-1}^j+6u_i^j-4u_{i+1}^j+u_{i+2}^j=\omega^4(\gamma u_{i-1}^{j(4)}+\beta u_i^{j(4)}+\gamma u_{i+1}^{j(4)})+lte_i^j\ ,i=2,3,...,n-2, \tag{14}$$

The relation (14) gives (n-2) algebraic equations in n unknowns. We need two more equations, one at each end of the range of the integration interval. Taylor series and the method of undetermined coefficients can be used to derive the two end conditions. The first end condition near x=0 is given by:

$$-2u_0^j+5u_1^j-4u_2^j+u_3^j=-\omega^2u_0^{j(2)}+\omega^4(\gamma_1u_o^{j(4)}+\gamma_2u_1^{j(4)}+\gamma_3u_2^{j(4)}+\gamma_4u_3^{j(4)})+lte_1^j, \quad (15)$$

The second one near *x=l* is given by:

$$-2u_n^j+5u_{n-1}^j-4u_{n-2}^j+u_{n-3}^j=-\omega^2u_n^{j(2)}+\omega^4(\gamma_1u_n^{j(4)}+\gamma_2u_{n-1}^{j(4)}+\gamma_3u_{n-2}^{j(4)}+\gamma_4u_{n-3}^{j(4)})+lte_n^j, \quad (16)$$

For the local truncation errors lte_i^j where $i=1,n,$ we have that:

$$lte_i^j=\begin{cases}O(h_t+\omega^4) & \text{for} \quad (\gamma_1,\gamma_2,\gamma_3,\gamma_4)=(2,6,3,0)/12.\\ O(h_t+\omega^8) & \text{for} \quad (\gamma_1,\gamma_2,\gamma_3,\gamma_4)=(28,245,56,1)/360.\end{cases}$$

Lemma 3 The local truncation errors $lte_i^j, i=2,3,...,n-2$ correlated with the scheme (14) are :

$$lte_i^j=(1-\beta-2\gamma)\omega^4D_x^4u_i^j+(\frac{1}{12}-\gamma)\omega^6D_x^6u_i^j+O(h_t+\omega^8), i=2,3,..,n-2, \ . \quad (17)$$

Proof To acquire the local truncation error, Taylor series expansion are used at the point (x_i,t_j).

Recall to our problem, we have

$$u_i^{j(4)}=D_x^4u_i^j=-D_t^{\alpha(t_j)}s_i^j-2D_x^2s_i^j-(1-\mu)s_i^j-f_i^j, \quad (18)$$

Using finite difference method, we get

$$D_x^2s_i^j=\frac{s_{i-1}^j-2s_i^j+s_{i+1}^j}{\omega^2}+O(\omega^2). \quad (19)$$

Using Eq. (5) and utilising that $\alpha(t_j)=\alpha_j$, we get:

$$D_t^{\alpha_j}s_i^j=\frac{1}{h_t^{\alpha_j}}\sum_{k=0}^{j}g_{\alpha_j,k}\,s_i^{j-k}, \quad (20)$$

where $g_{\alpha_j,k}, k=0,1,2,....j$ are the Grünwald-Letnikov weights,

$$g_{\alpha_j,k}=(1-\frac{\alpha_j+1)}{k})g_{\alpha_j,k-1}, \forall k\geq 2, g_{\alpha_j,0}=1 \text{ and } g_{\alpha_j,1}=-\alpha_j. \quad (21)$$

Substituting from Eqns. (20) and (19) into Eq. (18), we get

$$u_i^{j(4)}=\frac{-1}{h_t^{\alpha_j}}\sum_{k=0}^{j}g_{\alpha_j,k}\,s_i^{j-k}-\frac{2}{\omega^2}(s_{i-1}^j-2s_i^j+s_{i+1}^j)-(1-\mu)s_i^j-f_i^j. \quad (22)$$

Replace i by $i-1$ and $i+1$ in Eq. (22) respectively and then substitute in Eq. (14), we have the following consistency relation can be established:

$$\frac{\omega^4}{h_t^{\alpha_j}}(\gamma\sum_{k=0}^{j} g_{\alpha_j,k}\, s_{i-1}^{j-k} + \beta\sum_{k=0}^{j} g_{\alpha_j,k}\, s_i^{j-k} + \gamma\sum_{k=0}^{j} g_{\alpha_j,k}\, s_{i+1}^{j-k}) =$$

$$-s_{i-2}^j + 4s_{i-1}^j - 6s_i^j + 4s_{i+1}^j - s_{i+2}^j + \omega^4(\kappa s_{i-2}^j + \rho s_{i-1}^j + \delta s_i^j + \rho s_{i+1}^j + \kappa s_{i+2}^j)$$

$$-\omega^4(\gamma f_{i-1}^j + \beta f_i^j + \gamma f_{i+1}^j)\ ,\ i = 2,3,...,n-1 \text{ and } j = 1,2,...,m, \tag{23}$$

where

$$\kappa = \frac{-2\gamma}{\omega^2},\ \rho = \frac{4\gamma - 2\beta}{\omega^2} - \gamma(1-\mu) + \frac{\gamma\,\Gamma(1+\alpha_j)}{h_t^{\alpha_j}} \text{ and}$$

$$\delta = \frac{-4\gamma + 4\beta}{\omega^2} - \beta(1-\mu) + \frac{\beta\,\Gamma(1+\alpha_j)}{h_t^{\alpha_j}}. \tag{24}$$

Let $S^j = (s_i^j)$ and $U^j = (u_i^j)$ be the approximate and the exact solutions respectively. Then we can write the system given by Eqns. (15), (16) and (23) in matrix form as follows:

$$(A + \omega^4 C - \frac{\omega^4}{h_t^{\alpha_j}} B)U^j = \frac{\omega^4}{h_t^{\alpha_j}} B \sum_{k=1}^{j} g_{\alpha_j,k} U^{j-k} + \omega^4 B F^j,\ j = 1,2,...,m, \tag{25}$$

where $F^j = (f_i^j), i = 1,2,...,n$ and the matrices A, B and C are:

$$A = \begin{pmatrix} -5 & 4 & -1 & & & & \\ 4 & -6 & 4 & -1 & & & \\ -1 & 4 & -6 & 4 & -1 & & \\ & & & & \ddots & & \\ & & -1 & 4 & -6 & 4 & -1 \\ & & & -1 & 4 & -6 & 4 \\ & & & & -1 & 4 & -5 \end{pmatrix},\ B = \begin{pmatrix} \gamma_2 & \gamma_3 & \gamma_4 & & & & \\ \gamma & \beta & \gamma & & & & \\ & \gamma & \beta & \gamma & & & \\ & & & \ddots & & & \\ & & & \gamma & \beta & \gamma & \\ & & & & \gamma & \beta & \gamma \\ & & & & \gamma_4 & \gamma_3 & \gamma_2 \end{pmatrix}$$

and

$$C = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & & & & \\ \rho & \delta & \rho & \kappa & & & \\ & \kappa & \rho & \delta & \rho & \kappa & \\ & & & \ddots & & & \\ & \kappa & \rho & \delta & \rho & \kappa & \\ & & & \kappa & \rho & \delta & \rho \\ & & & & \sigma_3 & \sigma_2 & \sigma_1 \end{pmatrix}.$$

3 Stability analysis

In this section, the stability of our proposed scheme will be investigated via Fourier series method. We suppose that s_i^j, $i=1,2,3,...,n$ and $j=1,2,...,m$ be the approximate solution of Eq.(23) and for simplicity, we will rewrite this equation without the nonlinear term. This leads to

$$\gamma\sum_{k=0}^{j} g_{\alpha_j,k}\, s_{i-1}^{j-k} + \beta\sum_{k=0}^{j} g_{\alpha_j,k}\, s_{i}^{j-k} + \gamma\sum_{k=0}^{j} g_{\alpha_j,k}\, s_{i+1}^{j-k} =$$

$$\frac{h_t^{\alpha_j}}{\omega^4}[-s_{i-2}^j+4s_{i-1}^j-6s_i^j+4s_{i+1}^j-s_{i+2}^j]+h_t^{\alpha_j}(\kappa s_{i-2}^j+\rho s_{i-1}^j+\delta s_i^j+\rho s_{i+1}^j+\kappa s_{i+2}^j)$$

$$,\ i=2,3,...,n-1 \text{ and } j=1,2,...,m, \tag{26}$$

$$(\frac{-h_t^{\alpha_j}}{\omega^4}+h_t^{\alpha_j}\kappa)s_{i-2}^j+(\frac{4h_t^{\alpha_j}}{\omega^4}+h_t^{\alpha_j}\rho-\gamma)s_{i-1}^j+(\frac{-6h_t^{\alpha_j}}{\omega^4}+h_t^{\alpha_j}\delta-\beta)s_i^j+(\frac{4h_t^{\alpha_j}}{\omega^4}+h_t^{\alpha_j}\rho-\gamma)s_{i+1}^j$$

$$+(\frac{-h_t^{\alpha_j}}{\omega^4}+h_t^{\alpha_j}\kappa)s_{i+2}^j=\gamma\sum_{k=1}^{j} g_{\alpha_j,k}\, s_{i-1}^{j-k} + \beta\sum_{k=1}^{j} g_{\alpha_j,k}\, s_{i}^{j-k} + \gamma\sum_{k=1}^{j} g_{\alpha_j,k}\, s_{i+1}^{j-k},$$

$$,\ i=2,3,...,n-1 \text{ and } j=1,2,...,m, \tag{27}$$

Eq.(27) will be rewritten as:

$$a_1s_{i-2}^j+a_2s_{i-1}^j+a_3s_i^j+a_2s_{i+1}^j+a_1s_{i+2}^j=(\gamma\sum_{k=1}^{j} g_{\alpha_j,k}\, s_{i-1}^{j-k} + \beta\sum_{k=1}^{j} g_{\alpha_j,k}\, s_{i}^{j-k} + \gamma\sum_{k=1}^{j} g_{\alpha_j,k}\, s_{i+1}^{j-k}),$$

$$i=2,3,...,n-1 \text{ and } j=1,2,...,m, \tag{28}$$

where $\hbar=\frac{h_t^{\alpha_j}}{\omega^4}, a_1=(-\hbar+h_t^{\alpha_j}\kappa), a_2=(4\hbar+h_t^{\alpha_j}\rho-\gamma))$ and $a_3=(-6\hbar+h_t^{\alpha_j}\delta-\beta)$.

Lemma 4: [9]

For $0<\alpha(t)<1$, the coefficients $g_{\alpha(t),k}, k=0,1,2,...$ satisfies

(1) $g_{\alpha(t),0}=1, g_{\alpha(t),1}=-\alpha(t), g_{\alpha(t),k}<0$ for $k=1,2,...,$

(2) $\sum_{k=0}^{\infty} g_{\alpha(t),k}=0, \forall n\in N^+, -\sum_{k=1}^{n} g_{\alpha(t),k}<1.$

The round-off error is defined as:

$$\varepsilon_i^j=u_i^j-s_i^j \quad ,\ i=2,3,...,n-1 \text{ and } j=1,2,...,m \tag{29}$$

This error satisfies the Eq.(28), then we have:

$$a_1\varepsilon_{i-2}^j + a_2\varepsilon_{i-1}^j + a_3\varepsilon_i^j + a_2\varepsilon_{i+1}^j + a_1\varepsilon_{i+2}^j = \gamma\sum_{k=1}^{j} g_{\alpha_j,k}\ \varepsilon_{i-1}^{j-k} + \beta\sum_{k=1}^{j} g_{\alpha_j,k}\ \varepsilon_i^{j-k} + \gamma\sum_{k=1}^{j} g_{\alpha_j,k}\ \varepsilon_{i+1}^{j-k},$$

$$i = 2,3,\ldots,n-1 \text{ and } j = 1,2,\ldots,m, \tag{30}$$

Let us represent the error function $\varepsilon(i\,\omega) = \varepsilon_i, i = 0,1,2,\ldots,n$ as a Fourier series

$$\varepsilon_i = \sum_{k=0}^{n} A_k e^{\varpi q_k i\omega}, \varpi = \sqrt{-1}, i = 0,1,\ldots,n, \tag{31}$$

where $q_k = k\pi$.

For measuring the magnitude of the error vector $\varepsilon^j(x) = [\varepsilon_1^j, \varepsilon_2^j, \ldots, \varepsilon_{n-1}^j]^T$, $j = 0,1,2,\ldots,m$, we use the discrete l^2 norm of the form:

$$\left\|\varepsilon^j\right\|_{l^2} = \left(\sum_{i=1}^{n-1} \omega\left|\varepsilon_1^j\right|^2\right), j = 0,1,\ldots,m. \tag{32}$$

Assume that the solution of the error Eq.(30) has the form:

$$\varepsilon_i^j = \xi_j e^{\varpi q i\,\omega}, \xi_j = e^{\vartheta\pi t}, \tag{33}$$

and ϑ is a complex number.

Substituting by Eq. (33) into Eq.(30), we obtain

$$a_1\xi_j e^{\varpi q(i-2)\omega} + a_2\xi_j e^{\varpi q(i-1)\omega} + a_3\xi_j e^{\varpi qi\omega} + a_2\xi_j e^{\varpi q(i+1)\omega} + a_1\xi_j e^{\varpi q(i+2)\omega} =$$

$$(\gamma\sum_{k=1}^{j} g_{\alpha_j,k}\ \xi_{j-k} e^{\varpi q(i-1)\omega} + \beta\sum_{k=1}^{j} g_{\alpha_j,k}\ \xi_{j-k} e^{\varpi qi\,\omega} + \gamma\sum_{k=1}^{j} g_{\alpha_j,k}\ \xi_{j-k} e^{\varpi q(i+1)\omega})$$

$$, i = 2,3,\ldots,n-1 \text{ and } j = 1,2,\ldots,m. \tag{34}$$

By straightforward, this equation can be simplified into the form:

$$\xi_j(2a_1\cos 2q\omega + 2a_2\cos q\omega + a_3) = \sum_{k=1}^{j}(2\gamma\cos q\omega + \beta) g_{\alpha_j,k}\ \xi_{j-k},\ \ j = 1,2,\ldots,m, \tag{35}$$

From which we can accomplish that

$$\xi_j = \frac{(2\gamma\cos q\omega + \beta)}{(2a_1\cos 2q\omega + 2a_2\cos q\omega + a_3)}(-\alpha_j\xi_{j-1} + \sum_{k=2}^{j} g_{\alpha_j,k}\xi_{j-k}),\quad j = 1,2,\ldots,m, \tag{36}$$

Lemma 5

If ξ_j, $j = 1,2,\ldots,m$, satisfy Eq.(36), then we have $\left|\xi_j\right| \le \left|\xi_0\right|$.

Proof. Using the mathematical induction, for $j = 1$ in Eq.(36), then we acquire

$$\xi_1 = \frac{-\alpha_j (2\gamma \cos q\omega + \beta)}{(2a_1 \cos 2q\omega + 2a_2 \cos q\omega + a_3)} \xi_0$$

Then

$$|\xi_1| \le \left| \frac{-\alpha_j (2\gamma \cos q\omega + \beta)}{(2a_1 \cos 2q\omega + 2a_2 \cos q\omega + a_3)} \right| |\xi_0|$$

Substituting by the values of a_1, a_2 and a_3 we find that

$$(2a_1 \cos 2q\omega + 2a_2 \cos q\omega + a_3) = -(\beta + 2\gamma \cos q\omega)(1 + (1-\mu) h_t^{\alpha_j}$$

$$-\Gamma(1+\alpha_j) + \frac{8h_t^{\alpha_j}}{\omega^2} \sin^2(\frac{q\omega}{2})) - 8\hbar \sin^4(\frac{q\omega}{2}),$$

It can be confirmed that

$$\left| \frac{-\alpha_j (2\gamma \cos q\omega + \beta)}{(2a_1 \cos 2q\omega + 2a_2 \cos q\omega + a_3)} \right| < 1$$

Then

$$|\xi_1| \le |\xi_0|, \tag{37}$$

Now assume that $|\xi_n| \le |\xi_0|, n = 2, 3, ..., m$.

Now returning to Eq.(36) and using Lemma 4, we get

$$|\xi_j| \le \left| \frac{(2\gamma \cos q\omega + \beta)}{(2a_1 \cos 2q\omega + 2a_2 \cos q\omega + a_3)} \right| (\alpha_j |\xi_{j-1}| + \sum_{k=2}^{j} |g_{\alpha_j,k}| |\xi_{j-k}|), \quad j = 1, 2, ..., m,$$

$$|\xi_j| \le \left| \frac{(2\gamma \cos q\omega + \beta)}{(2a_1 \cos 2q\omega + 2a_2 \cos q\omega + a_3)} \right| (\alpha_j + \sum_{k=2}^{j} |g_{\alpha_j,k}|) |\xi_0|$$

$$\le \left| \frac{(2\gamma \cos q\omega + \beta)}{(2a_1 \cos 2q\omega + 2a_2 \cos q\omega + a_3)} \right| (\alpha_j + (-\sum_{k=1}^{j} |g_{\alpha_j,k}| - \alpha_j)) |\xi_0|$$

$$\le \left| \frac{(2\gamma \cos q\omega + \beta)}{(2a_1 \cos 2q\omega + 2a_2 \cos q\omega + a_3)} \right| (\alpha_j + (1-\alpha_j)) |\xi_0|$$

$$\le \left| \frac{(2\gamma \cos q\omega + \beta)}{(2a_1 \cos 2q\omega + 2a_2 \cos q\omega + a_3)} \right| |\xi_0|.$$

Then we accomplish that $|\xi_j| \le |\xi_0|, j = 1, 2, 3, ..., m$.

Theorem 1

The spline solution of the Eqns. (1-2) defined by Eq. (23) is unconditionally stable.

Proof

Using Eq.(32) and Lemma 5, we achieve that

$$\left\|\varepsilon^{j}\right\|_{l^2} \le \left\|\varepsilon^{0}\right\|_{l^2}, \ j = 1,2,3,\ldots,m.$$

This completes the proof.

4 Convergence

The numerical scheme for the fractional partial differential equation is convergent of order *p* if and only if it is stable and consistent of order *p* [9]. In this section, we will discuss the convergence of our proposed scheme by using Fourier series method.

We suppose that $u(x_i, t_j)$ be the exact solution of Eq. (23) represented by Taylor series, then from Lemma 3 and using $(\gamma, \beta) = (1,10)/12$, we acquire that the truncation error is:

$$T_i^j = O(h_t, \omega^4).$$

Then, there is a positive constant c_1 depends on the analytical solution $u(x_i, t_j)$ such that:

$$\left|T_i^j\right| \le c_1(h_t + \omega^4), \tag{38}$$

The error is defined as follows:

$$e_i^j = u(x_i, t_j) - u_i^j \quad , \ i = 2,3,\ldots,n-1 \text{ and } j = 1,2,\ldots,m, \tag{39}$$

Then we get:

$$\begin{aligned}
&(\gamma \sum_{k=0}^{j} g_{\alpha_j,k} u(x_{i-1}, t_{j-k}) + \beta \sum_{k=0}^{j} g_{\alpha_j,k} u(x_i, t_{j-k}) + \gamma \sum_{k=0}^{j} g_{\alpha_j,k} u(x_{i+1}, t_{j-k})) \\
&= \hbar[-u(x_{i-2}, t_j) + 4u(x_{i-1}, t_j) - 6u(x_i, t_j) + 4u(x_{i+1}, t_j) - u(x_{i+2}, t_j)] \\
&\quad + h_t^{\alpha_j} (\kappa u(x_{i-2}, t_j) + \rho u(x_{i-1}, t_j) + \delta u(x_i, t_j) + \rho u(x_{i+1}, t_j) + \kappa u(x_{i+2}, t_j)) \\
&\quad - h_t^{\alpha_j} (\gamma \tilde{f}_{i-1}^{j} + \beta \tilde{f}_i^{j} + \gamma \tilde{f}_{i+1}^{j}) - T_i^j, \ i = 2,3,\ldots,n-1 \text{ and } j = 1,2,\ldots,m,
\end{aligned} \tag{40}$$

where $\tilde{f}_i^j = f(u(x_i, t_j))$.

To get the error equation, we subtract Eq. (23) from Eq. (40), then we achieve:

$$\begin{aligned}
&(\gamma \sum_{k=0}^{j} g_{\alpha_j,k} e_{i-1}^{j-k} + \beta \sum_{k=0}^{j} g_{\alpha_j,k} e_i^{j-k} + \gamma \sum_{k=0}^{j} g_{\alpha_j,k} e_{i+1}^{j-k}) \\
&= \hbar[-e_{i-2}^j + 4e_{i-1}^j - 6e_i^j + 4e_{i+1}^j - e_{i+2}^j] + h_t^{\alpha_j} (\kappa e_{i-2}^j + \rho e_{i-1}^j + \delta e_i^j + \rho e_{i+1}^j + \kappa e_{i+2}^j)
\end{aligned}$$

$$-h_t^{\alpha_j}(\gamma\tilde{e}_{i-1}^j+\beta\tilde{e}_i^j+\gamma\tilde{e}_{i+1}^j)-T_i^j,\ i=2,3,...,n-1 \text{ and } j=1,2,...,m, \tag{41}$$

where $\tilde{e}_i^j=\tilde{f}_i^j-f_i^j$ and from condition (3) we get that $\tilde{e}_i^j\le\theta e_i^j$, then the error equation will be

$$(\gamma\sum_{k=0}^{j}g_{\alpha_j,k}e_{i-1}^{j-k}+\beta\sum_{k=0}^{j}g_{\alpha_j,k}e_i^{j-k}+\gamma\sum_{k=0}^{j}g_{\alpha_j,k}e_{i+1}^{j-k})$$
$$=\hbar[-e_{i-2}^j+4e_{i-1}^j-6e_i^j+4e_{i+1}^j-e_{i+2}^j]+h_t^{\alpha_j}(\kappa e_{i-2}^j+\rho e_{i-1}^j+\delta e_i^j+\rho e_{i+1}^j+\kappa e_{i+2}^j)$$
$$-h_t^{\alpha_j}\theta(\gamma e_{i-1}^j+\beta e_i^j+\gamma e_{i+1}^j)-T_i^j,\ i=2,3,...,n-1 \text{ and } j=1,2,...,m, \tag{42}$$

with error boundary conditions , $e_0^j=e_n^j=0,\ j=1,2,...,m$.

Eq.(42) can be rearranged as in the previous section, and it will take the form:

$$a_1e_{i-2}^j+a_2e_{i-1}^j+a_3e_i^j+a_2e_{i+1}^j+a_1e_{i+2}^j=(\gamma\sum_{k=1}^{j}g_{\alpha_j,k}e_{i-1}^{j-k}+\beta\sum_{k=1}^{j}g_{\alpha_j,k}e_i^{j-k}+\gamma\sum_{k=1}^{j}g_{\alpha_j,k}e_{i+1}^{j-k})$$
$$+h_t^{\alpha_j}\theta(\gamma e_{i-1}^j+\beta e_i^j+\gamma e_{i+1}^j)+T_i^j,\ i=2,3,...,n-1 \text{ and } j=1,2,...,m, \tag{43}$$

We define the discrete functions

$$e^j(x)=\begin{cases}e_i^j & \text{when } x_{i-\omega/2}<x\le x_{i+\omega/2},\ i=1,2,...,n-1,\\ 0 & \text{when } 0\le x\le\omega/2 \text{ or } 1-\omega/2<x\le1,\end{cases}$$

and

$$T^j(x)=\begin{cases}T_i^j & \text{when } x_{i-\omega/2}<x\le x_{i+\omega/2},\ i=1,2,...,n-1,\\ 0 & \text{when } 0\le x\le\omega/2 \text{ or } 1-\omega/2<x\le1,\end{cases}$$

$e^j(x)$ and $T^j(x)$ can be expanded in Fourier series as

$$e^j(x)=\sum_{k=-\infty}^{\infty}\lambda_j(k)e^{2\varpi\pi kx},\ j=0,1,...,m, \tag{44}$$

$$T^j(x)=\sum_{k=-\infty}^{\infty}\eta_j(k)e^{2\varpi\pi kx},\ j=0,1,...,m, \tag{45}$$

where

$$\lambda_j(k)=\int_0^1 e^j(x)e^{-2\varpi\pi kx}dx, \tag{46}$$

$$\eta_j(k)=\int_0^1 T^j(x)e^{-2\varpi\pi kx}dx, \tag{47}$$

From definition of l^2 norm and Parseval equality, we get

$$\left\|e^j\right\|_{l^2}^2 = \sum_{i=1}^{n-1}\omega\left|e_i^j\right|^2 = \int_0^1\left|e^j(x)\right|^2 dx = \sum_{k=-\infty}^{\infty}\left|\lambda_j(k)\right|^2,\ j=0,1,...,m, \tag{48}$$

and

$$\left\|T^j\right\|_{l^2}^2 = \sum_{i=1}^{n-1}\omega\left|T_i^j\right|^2 = \int_0^1\left|T^j(x)\right|^2 dx = \sum_{k=-\infty}^{\infty}\left|\eta_j(k)\right|^2,\ j=0,1,...,m, \tag{49}$$

Assume that

$$e_i^j = \lambda_j e^{\varpi\varsigma i\omega}, \tag{50}$$

and

$$T_i^j = \eta_j e^{\varpi\varsigma i\omega}, \tag{51}$$

where $\varsigma = 2\pi k$.

Substituting by Eqs. (50) and (51) into Eq.(43), we obtain:

$$\lambda_j = \frac{1}{(2a_1\cos 2\varsigma\omega + 2a_2\cos\varsigma\omega + a_3)}((2\gamma\cos\varsigma\omega + \beta)[(-\alpha_j + \omega^4\theta)\lambda_{j-1} + \sum_{k=2}^{j} g_{\alpha_j,k}\lambda_{j-k}] + \eta_j),$$

$$j = 1,2,...,m, \tag{52}$$

and

$$(2a_1\cos 2\varsigma\omega + 2a_2\cos\varsigma\omega + a_3) = -(\beta + 2\gamma\cos\varsigma\omega)(1+(1-\mu)h_t^{\alpha_j}$$

$$-\Gamma(1+\alpha_j) + \frac{8h_t^{\alpha_j}}{\omega^2}\sin^2(\frac{\varsigma\omega}{2})) - 8\hbar\sin^4(\frac{\varsigma\omega}{2}), \tag{53}$$

Lemma 6

If $\lambda_{j+1}, (j=0,1,...,m)$ satisfy Eq.(52) then $\left|\lambda_{j+1}\right| \le (j+1)c_3\left|\eta_1\right|$,where c_3 is a positive constant and $j=0,1,...,m$.

Proof

From Eq.(38) and Eq.(49), we get

$$\left\|T^j\right\|_{l^2} \le c_1\sqrt{n\omega}(h_t + \omega^4)$$

$$\le c_2(h_t + \omega^4)\ ,\ j=0,1,...,m, \tag{54}$$

where $c_2 = c_1\sqrt{n\omega} = c_1 l$.

From Eqs.(47) and (49), there is a positive constant c_3 so that

$$\left|\eta_j\right| < c_3\left|\eta_1\right| \; j=0,1,2,...,m\,, \tag{55}$$

and we have that

$$\lambda_0 = 0\,. \tag{56}$$

Using the mathematical induction, from Eq.(52), for $j=1$ we get :

$$\lambda_1 = \frac{1}{(2a_1\cos 2\varsigma\omega + 2a_2\cos\varsigma\omega + a_3)}\eta_0,$$

From Eqs.(53) and (55) we can find that;

$$\left|\lambda_1\right| \le \left|\eta_0\right| \le c_3\left|\eta_0\right|,$$

Now suppose that

$$\left|\lambda_j\right| \le c_3\, j\left|\eta_0\right|, \quad j=1,2,...,m, \tag{57}$$

From Eqns (52) and (55) and Lemma 4 and by the same way as in the previous section we can find that:

$$\left|\lambda_{j+1}\right| \le \left|\frac{1}{(2a_1\cos 2\varsigma\omega + 2a_2\cos\varsigma\omega + a_3)}\right| [(2\gamma\cos\varsigma\omega + \beta)[(\alpha_j + \omega^4\theta)\left|\lambda_j\right| + \sum_{k=2}^{j}\left|g_{\alpha_j,k}\right|\left|\lambda_{j-k+1}\right|] + \left|\eta_{j+1}\right|],$$

$$\le \left|\frac{1}{(2a_1\cos 2\varsigma\omega + 2a_2\cos\varsigma\omega + a_3)}\right| [(2\gamma\cos\varsigma\omega + \beta)[(\alpha_j + \omega^4\theta)j + \sum_{k=2}^{j}\left|g_{\alpha_j,k}\right|(j-k+1)] + 1]c_3\left|\eta_1\right|$$

$$\le \left|\frac{1}{(2a_1\cos 2\varsigma\omega + 2a_2\cos\varsigma\omega + a_3)}\right| [(2\gamma\cos\varsigma\omega + \beta)[(\alpha_j + \omega^4\theta)j + (-\sum_{k=1}^{j} g_{\alpha_j,k} - \alpha_j)j + 1]c_3\left|\eta_1\right|$$

$$\le \left|\frac{1}{(2a_1\cos 2\varsigma\omega + 2a_2\cos\varsigma\omega + a_3)}\right| [(2\gamma\cos\varsigma\omega + \beta)[(\alpha_j + \omega^4\theta)j + (1-\alpha_j)j] + 1]c_3\left|\eta_1\right|$$

$$\le \left|\frac{1}{(2a_1\cos 2\varsigma\omega + 2a_2\cos\varsigma\omega + a_3)}\right| [(2\gamma\cos\varsigma\omega + \beta)(1+\omega^4\theta)j + 1]c_3\left|\eta_1\right|$$

$$\left|\lambda_{j+1}\right| \le (j+1)c_3\left|\eta_1\right|, \tag{58}$$

Theorem 2

The spline solution of the Eqns (1-2) defined by Eq. (23) is convergent, and the order of convergence is $O(h_t + \omega^4)$.

Proof: From Eqns.(48), (49), (50) and Lemma 6, we can find that

$$\left\|e^{j+1}\right\|_{l^2} \le (j+1)c_3\left\|T^1\right\|_{l^2} \le (j+1)c_3c_2(h_t + \omega^4)$$

$$\le c(h_t + \omega^4),$$

where $c=(j+1)c_3c_2$. This completes the proof.

Numerical Results:

In this section, the numerical results of $u(x,t)$ for the non-linear time fractional Swift Hohenberg equation has been achieved for the one-dimensional domain for various values of fractional Brownian α as a function of time subjected to $0<\alpha(t)\leq 1$ and also the standard motion $\alpha=1$. The effect of the parameters and length of the domain on the results are presented graphically.

In the following, the approximate solutions $u(x,t)$ for various values of α as a function of time using $(\gamma_1,\gamma_2,\gamma_3,\gamma_4)=(2,6,3,0)/12$ are represented in Fig.1 and Fig.2. Figs (4) and (4) represent the approximate solutions $u(x,t)$ using $(\gamma_1,\gamma_2,\gamma_3,\gamma_4)=(28,245,56,1)/360$. Fig.(4) displays the approximate solutions $u(x,t)$ for the standard motion $\alpha=0.5$ and $\alpha=1$.

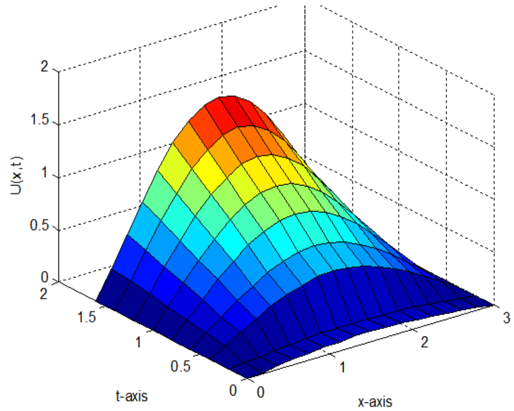

Fig.(1-a), $l = 3, \mu = 0.3$ and $\alpha(t) = 0.5t$

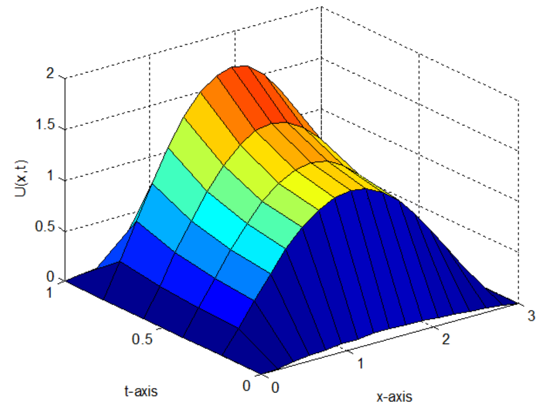

Fig.(1-b), $l = 3, \mu = 0.3$ and $\alpha(t) = t^2$

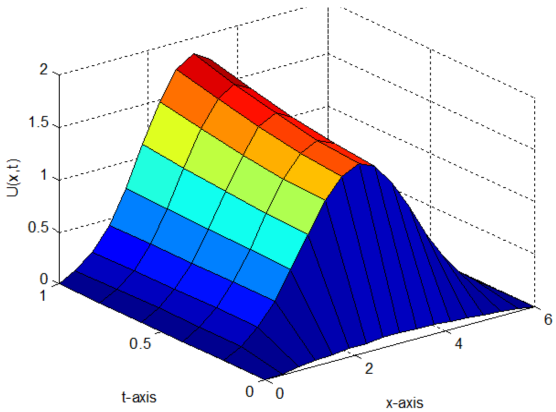

Fig.(1-c), $l = 6, \mu = 0.3$ and $\alpha(t) = 0.5t$

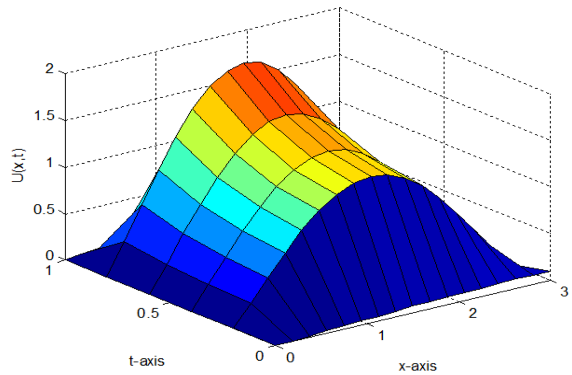

Fig.(1-d), $l = 6, \mu = 0.3$ and $\alpha(t) = t^2$

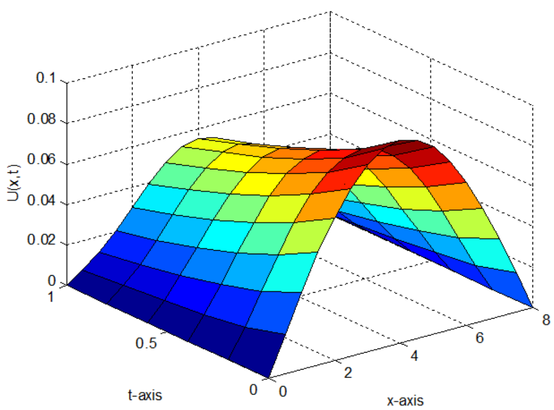

Fig.(1-e), $l = 8, \mu = 0.3$ and $\alpha(t) = 1 - 0.1\sin t$

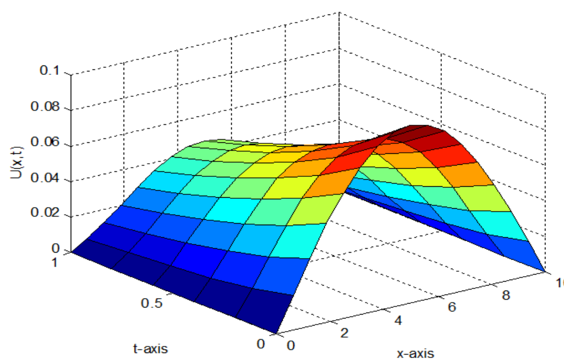

Fig.(1-f), $l = 10, \mu = 0.3$ and $\alpha(t) = 1 - 0.1\sin t$

Fig.1, Approximate solutions $u(x,t)$ with $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (2,6,3,0)/12$.

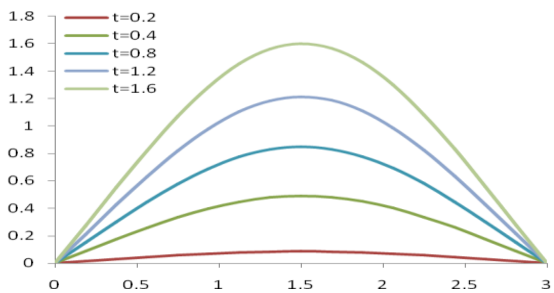

Fig.(2-c), $l = 3, \mu = 0.3$ and $\alpha(t) = 0.5t$

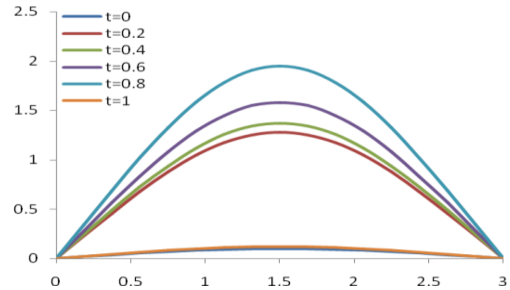

Fig.(2-d), $l = 3, \mu = 0.3$ and $\alpha(t) = t^2$

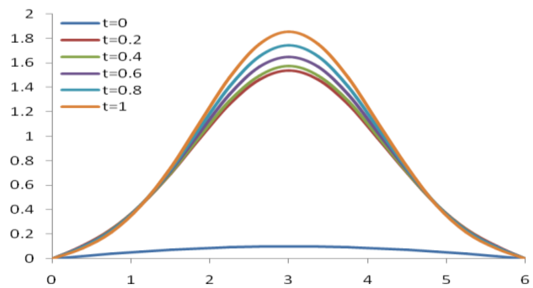

Fig.(2-e), $l = 6, \mu = 0.3$ and $\alpha(t) = 0.5t$

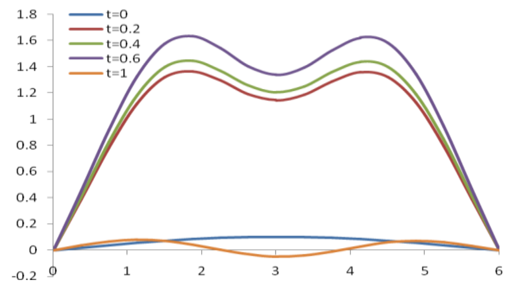

Fig.(2-f), $l = 6, \mu = 0.3$ and $\alpha(t) = t^2$

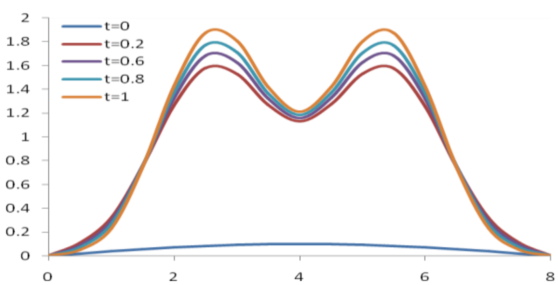

Fig.(2-e), $l = 8, \mu = .3$ and $\alpha(t) = 0.5t$

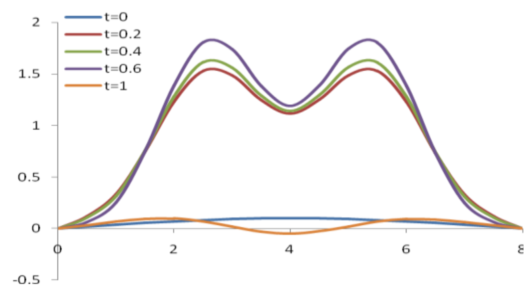

Fig.(2-e), $l = 8, \mu = .3$ and $\alpha(t) = t^2$

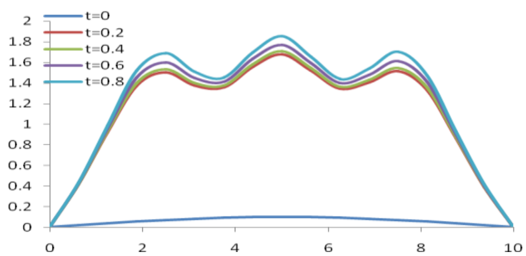

Fig.(2-a), $l = 10, \mu = .3$ and $\alpha(t) = 0.5t$

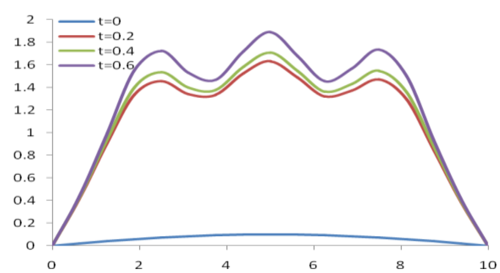

Fig.(2-b), $l = 10, \mu = 0.3$ and $\alpha(t) = t^2$

Fig2, Approximate solutions $u(x,t)$ with $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (2,6,3,0)/12$.

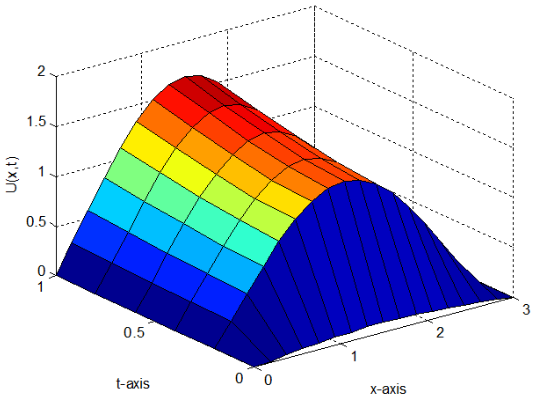

Fig.(3-a), $l = 3, \mu = 0.3$ and $\alpha(t) = 0.5t$

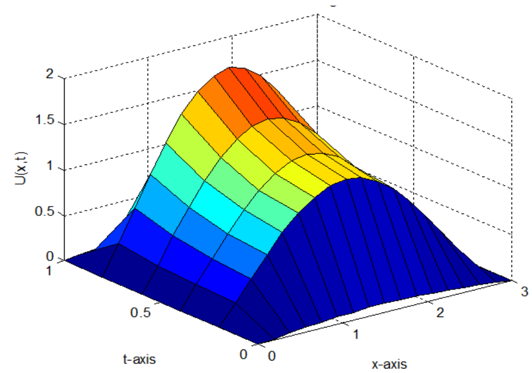

Fig.(3-b), $l = 3, \mu = 0.3$ and $\alpha(t) = t^2$

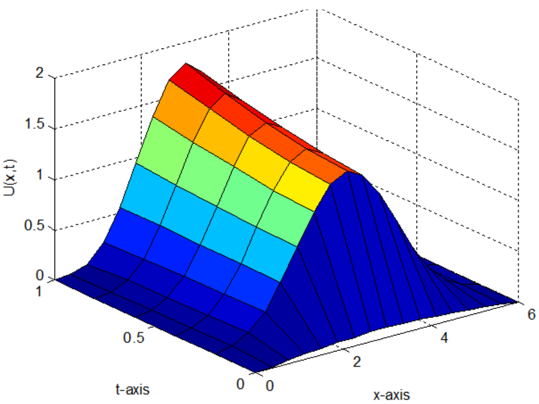

Fig.(3-c), $l = 6, \mu = 0.3$ and $\alpha(t) = 0.5t$

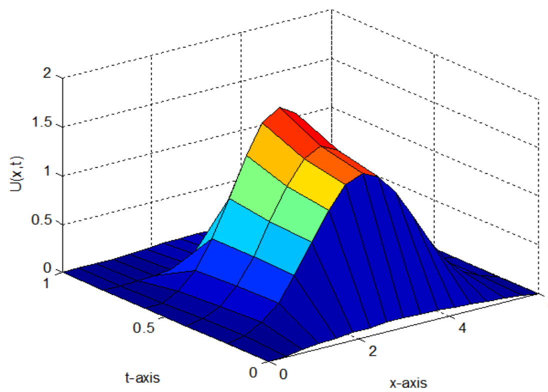

Fig.(3-d), $l = 6, \mu = 0.3$ and $\alpha(t) = t^2$

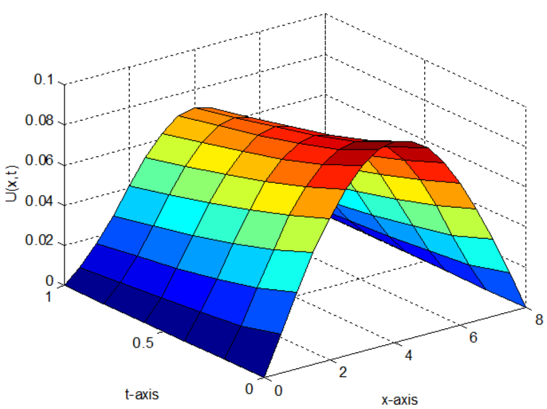

Fig.(3-e), $l = 8, \mu = 0.3$ and $\alpha(t) = 1 - 0.1\sin t$

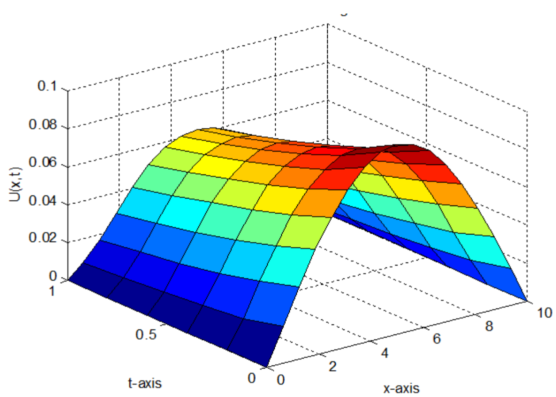

Fig.(3-f), $l = 10, \mu = 0.3$ and $\alpha(t) = 1 - 0.1\sin t$

Fig.3, Approximate solutions $u(x,t)$ with $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (28, 245, 56, 1)/360$.

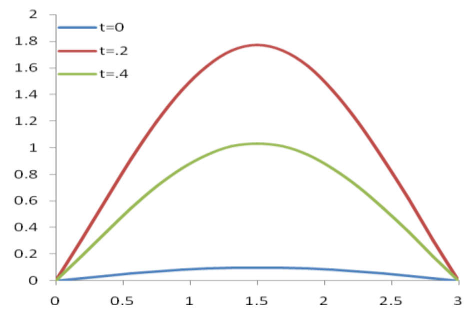

Fig.(4-a), $l = 3, \mu = .3$ and $\alpha = 0.5$

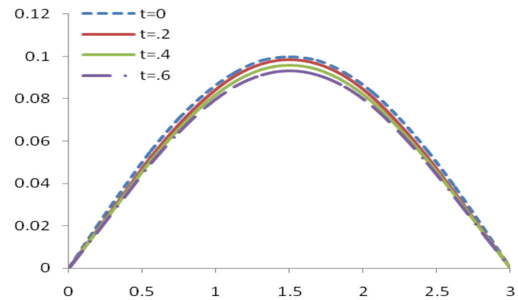

Fig.(4-b), $l = 3, \mu = 0.3$ and $\alpha = 1$

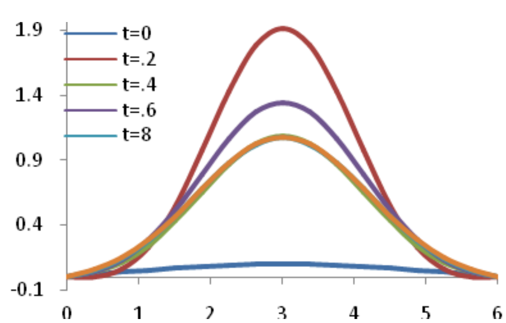

Fig.(4-c), $l = 6, \mu = 0.3$ and $\alpha = 0.5$

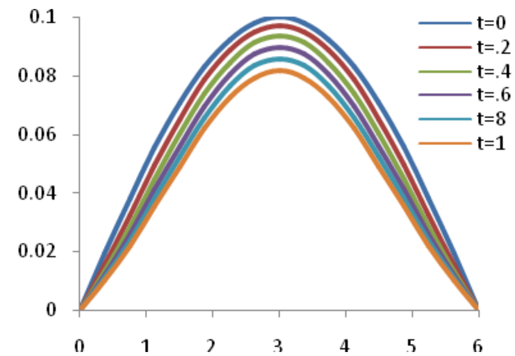

Fig.(4-d), $l = 6, \mu = 0.3$ and $\alpha = 1$

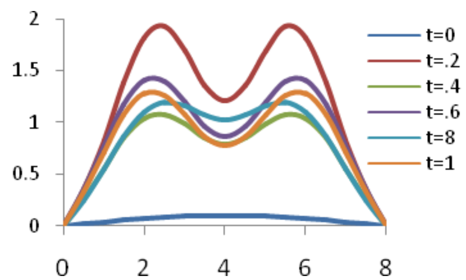

Fig.(4-e), $l = 8, \mu = 0.3$ and $\alpha = 0.5$

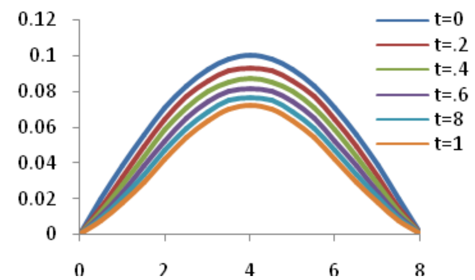

Fig.(4-f), $l = 8, \mu = 0.3$ and $\alpha = 1$

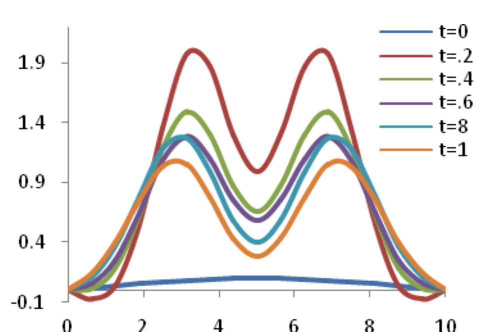

Fig.(4-g), $l = 10, \mu = 0.3$ and $\alpha = 0.5$

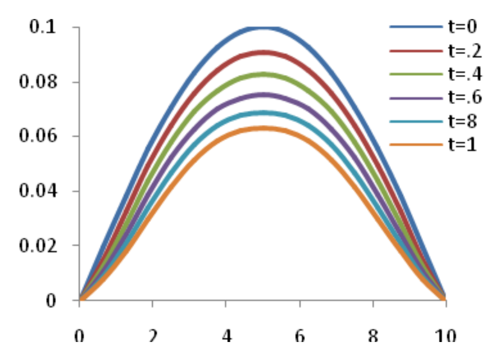

Fig.(4-h), $l = 10, \mu = 0.3$ and $\alpha = 1$

Fig.4, Approximate solutions $u(x,t)$ using $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (28, 245, 56, 1)/360$.

These figures demonstrate the effects of μ and l on the solutions as illustrated in resulted figures. Fig.1, Fig.2 and Fig.3 express the effect of time varying fractional order $\alpha(t)$, on the solutions. Comparing the obtained results plotted in Figs 1 and 3, it can be found that there is a relatively varying in the results according to the used scheme. It can be observed that our obtained results for $\alpha = 1$ and $\alpha = 0.5$ are matched with that published in [16, 18, 21-22]. All calculations are implemented with MATLAB R2015a.

Conclusion

In this paper, discrete spline function is used to approximate solution of variable time-fractional Swift-Hohenberg equation in the sense of Riemann Liouville derivative. The scheme is tested for some cases. The results demonstrate the reliability and efficiency of the algorithm developed. Stability and convergence analysis of the methods are presented. For the illustration of the practical usefulness of the proposed methods, some numerical results are included.

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