ADVANCED APPROACH FOR MULTI-OBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS

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Abstract. In this paper, a new approach for finding all efficient solutions for multiobjective fractional programming problems is presented. This approach based on solving auxiliary problems, one of them to obtain minimizing the numerator and the other maximizing the denominator. Illustrative examples are presented to clarify the obtained results.

Keywords. Efficient solution**,** Multi-objective fractional programming, Convex multi-objective programming.

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1. Introduction

Fractional programming problem is that in which the objective function is the ratio of the numerator and denominator. These types of problems have attracted considerable research and interest. Since these are useful in production planning, financial and corporate planning, health care and hospital planning etc.
There are different solution algorithms for determining the optimal solution of particular kinds of fractional programming problems. For example, Charnes and Cooper [2], Isbell and Marlow [6], Martos [8] and Wolf [13] solves linear fractional programming. Integer linear fractional programming has been solved by Rajendra [10], Seshan and Tibekar [11], Chandra and Chandramoham [1]. Swarup [12] gives an algorithm for solving quadratic fractional programming. The case where the restrictions are linear and the objective function is the quotient of a convex function with a concave function is solved by Mangasarian [7] using Frank and Wolf`s algorithm [5]. Dinkelbach [3] also considered the same objective over a convex feasible set. He solved this problem by solving a sequence of non-linear convex programming problems. Finally, E. A. Youness [14] presented a two dimensional approach for finding solutions of nonlinear fractional programming problems.

Multi-objective optimization problems are a class of optimization problems in which several different objective functions have to be considered simultaneously. Usually, there is no solution optimizing simultaneously all the several objective functions. Therefore, we search the so-called efficient solutions. When all the objective functions and the constraint functions

forming the feasible region are linear, then the multi-objective optimization problem is called linear. If at least one of the objective or the constraint functions is nonlinear, the problem is called a nonlinear multi-objective optimization problem. The multi-objective optimization problem is convex if all the objective functions and the feasible region are convex [9].
In 1989 Ebrahim. A. Youness [4] presented an approach for solving multi-objective fractional programming problems.

In this paper, a new approach for finding all efficient solutions for multi-objective fractional programming problems is presented. Illustrative examples are presented to clarify the obtained results.

2. Problem formulation

Consider the following multi-objective programming problem:

$$P_1 \begin{cases} Min \dfrac{\phi(x)}{\psi(x)} \\ s.t \\ M = \left\{X \in R^n : g_r(x) \le 0, r = 1,2,\dots,m\right\}, \end{cases}$$

where $\phi(x) = (\phi_1, \phi_2, \dots, \phi_m)$, $g_r(x), r = 1,2,\dots,m$ are convex functions and $\psi(x) = (\psi_1, \psi_2, \dots, \psi_m)$, $\psi(x) > 0$ is a concave function, and $M \subset R^n$, $M \ne \phi$ is the constraint set.

Definition: $x^* \in M$ is said to be an efficient solution for P_1 if there is no $x \in M$ such that

$$\frac{\phi(x)}{\psi(x)} \le \frac{\phi(x^*)}{\psi(x^*)}, \ \frac{\phi(x)}{\psi(x)} \ne \frac{\phi(x^*)}{\psi(x^*)}.$$

To find the efficient solution of problem P_1, formulate two auxiliary problems P_2 and P_3, then construct the efficient solution of P_1 from the efficient solutions of P_2 and P_3.

Now, consider the auxiliary problems P_2 and P_3 as following:

$$P_2 \begin{cases} Min \ \phi(x) \\ s.t \\ M = \left\{X \in R^n : g_r(x) \le 0, r = 1,2,\dots,m\right\} \end{cases}, \quad P_3 \begin{cases} Max \ \psi(x) \\ s.t \\ M = \left\{X \in R^n : g_r(x) \le 0, r = 1,2,\dots,m\right\} \end{cases}$$

Let the set of all efficient solutions of P_2 and P_3 is M_1^* and M_2^* respectively defined as:

$$M_1^* = \{\bar{x} : \bar{x} \in M\}, \ M_2^* = \{\bar{\bar{x}} : \bar{\bar{x}} \in M\}$$

3. Main results

The basic idea of this study is based on constructing the set of all efficient solution $M^* = \{x^* : x^* \in M\}$ for the problem P_1 from the efficient solutions M_1^* and M_2^* for the two auxiliary problems P_2 and P_3 as following:

Lemma 3.1. If $M_1^* \subset M_2^*$, then $\bar{x}$ is the efficient solution of P_1, $M^* = M_1^* \cap M_2^* = M_1^*$.

Proof.

Since $\bar{x} \in M_1^*$, $M_1^* \subset M_2^*$,then $\bar{x} \in M_2^*$. Thus

$$\psi(x) \le \psi(\bar{x}) \le \psi(\bar{\bar{x}}) \quad \forall x \in M \qquad (1)$$

Since $\bar{x}$ is the efficient solution of P_2, then

$$\phi(\bar{x}) \le \phi(x) \quad \forall x \in M \qquad (2)$$

Multiple (2) on $\psi(\bar{x})$, we get

$$\phi(\bar{x}).\psi(\bar{\mathrm{x}}) \le \phi(x).\psi(\bar{\mathrm{x}})$$

$$\frac{\phi(\bar{x})}{\psi(\bar{\mathrm{x}})} \le \frac{\phi(x)}{\psi(\bar{\mathrm{x}})} \qquad (3)$$

From (1), we get

$$\frac{1}{\psi(\bar{\mathrm{x}})} \le \frac{1}{\psi(\mathrm{x})} \qquad (4)$$

From (4) on (3), we get

$$\frac{\phi(\bar{x})}{\psi(\bar{\mathrm{x}})} \le \frac{\phi(x)}{\psi(\bar{\mathrm{x}})} \le \frac{\phi(x)}{\psi(\mathrm{x})}$$

So,

$$\frac{\phi(\bar{x})}{\psi(\bar{\mathrm{x}})} \le \frac{\phi(x)}{\psi(\mathrm{x})}$$

Which mean that the efficient solution for P_1 is $\bar{x}$, $\bar{x} \in M^* = M_1^* \cap M_2^*$.

Lemma 3.2. If $M_2^* \subset M_1^*$, then $\bar{x}$ is the efficient solution of P_1, $M^* = M_1^* \cup M_2^* = M_1^*$.

Proof.

Since $\bar{\bar{x}} \in M_2^*$, $M_2^* \subset M_1^*$,then $\bar{\bar{x}} \in M_1^*$. Thus

$$\phi(\bar{x}) \le \phi(\bar{\bar{x}}) \le \phi(x) \quad \forall x \in M \qquad (1)$$

Then

$$\phi(\bar{x}) \le \phi(\bar{\bar{x}}) \qquad (2)$$

Multiple (2) on $\psi(\bar{x})$, we get

$$\phi(\bar{x}).\psi(\bar{x}) \le \phi(\bar{\bar{x}}).\psi(\bar{x})$$

$$\frac{\phi(\bar{x})}{\psi(\bar{x})} \le \frac{\phi(\bar{\bar{x}})}{\psi(\bar{x})} \qquad (3)$$

Since $\bar{\bar{x}}$ is the efficient solution of P_3, then

$$\psi(x) \le \psi(\bar{x}) \le \psi(\bar{\bar{x}}) \quad \forall x \in M$$

$$\frac{1}{\psi(\bar{\bar{x}})} \le \frac{1}{\psi(\bar{x})} \le \frac{1}{\psi(x)} \qquad (4)$$

From (1), (4) on (3), we get

$$\frac{\phi(\bar{x})}{\psi(\bar{x})} \le \frac{\phi(\bar{\bar{x}})}{\psi(\bar{x})} \le \frac{\phi(\bar{\bar{x}})}{\psi(x)} \le \frac{\phi(x)}{\psi(x)}$$

So,

$$\frac{\phi(\bar{x})}{\psi(\bar{x})} \le \frac{\phi(x)}{\psi(x)}$$

Which mean that the efficient solution for P_1 is $\bar{x},\ \bar{x} \in M^* = M_1^* \cup M_2^*$.

Lemma 3.3. If $M_1^* \cap M_2^* = \phi$, then the efficient solution of P_1 is $x^* \in M^*$, $x^* \notin M_1^*,\ x^* \notin M_2^*,\ M_1^* \le M^* \le M_2^*$.

Proof.

Since $x^* \notin M_1^*,\ x^* \notin M_2^*$, then

$$\phi(\bar{x}) \le \phi(x^*) \quad \forall x^* \in M \tag{1}$$

$$\psi(x^*) \le \psi(\bar{\bar{x}}) \qquad \forall x^* \in M$$

$$\frac{1}{\psi(\bar{\bar{x}})} \le \frac{1}{\psi(x^*)} \tag{2}$$

Multiple (2) on $\phi(\bar{x})$, we get

$$\frac{\phi(\bar{x})}{\psi(\bar{\bar{x}})} \le \frac{\phi(\bar{x})}{\psi(x^*)} \tag{3}$$

From (1) on (3), we get

$$\frac{\phi(\bar{x})}{\psi(\bar{\bar{x}})} \le \frac{\phi(\bar{x})}{\psi(x^*)} \le \frac{\phi(x^*)}{\psi(x^*)} \tag{4}$$

So,

$$\frac{\phi(\bar{x})}{\psi(\bar{\bar{x}})} \le \frac{\phi(x^*)}{\psi(x^*)} \le 1$$

$$\phi(\bar{x}) \le \frac{\phi(x^*)}{\psi(x^*)} \le \psi(\bar{\bar{x}})$$

Which mean that the efficient solution for P_1 is $x^*,\ x^* \in M^*,\ M_1^* \le M^* \le M_2^*$.

Lemma 3.4. Let M is convex set and $\phi(x)$ convex on M , then if $\bar{x}$ is efficient solution for P_1, then $\bar{x}$ is efficient solution for P_2 .

Proof.

Let $\bar{x}$ not efficient solution for P_2, then there exist $x^{**} \in M$ such that $\phi(x^{**}) \le \phi(\bar{x})$, $\phi(x^{**}) \ne \phi(\bar{x})$.

Since M is a convex set, then from convexity there exist $\hat{x} \in M$, such that

$$\hat{x} = (1-\lambda)\ \bar{x} + \lambda\ x^{**},\ 0 \le \lambda \le 1$$

Let $\phi(x)$ is a convex set, then from convexity, we get

$$\phi(\hat{x}) \le (1-\lambda)\ \phi(\bar{x}) + \lambda\ \phi(x^{**}),$$

at $\lambda = 0$, we get

$$\phi(\hat{x}) \le \phi(\bar{x})$$

Which contradict the assumption, hence $\bar{x}$ efficient solution for P_2.

Lemma 3.5. Let M is convex set and $\psi(x)$ concave on M , then if $\bar{x}$ is efficient solution for P_1, then $\bar{x}$ is efficient solution for P_3 .

Proof.

Let $\bar{x}$ not efficient solution for P_3, then there exist $x^{**} \in M$ such that $\psi(x^{**}) \ge \psi(\bar{x})$, $\psi(x^{**}) \ne \psi(\bar{x})$.

Since M is a convex set, then from convexity there exist $\hat{x} \in M$, such that

$$\hat{x} = (1-\lambda)\,\bar{x} + \lambda\, x^{**},\ \ 0 \le \lambda \le 1$$

Let $\psi(x)$ is a convex set, then from convexity, we get

$$\psi(\hat{x}) \ge (1-\lambda)\ \psi(\bar{x}) + \lambda\ \psi(x^{**}),$$

at $\lambda = 0$, we get

$$\psi(\hat{x}) \ge \psi(\bar{x})$$

Which contradict the assumption, hence $\bar{x}$ efficient solution for P_3.

Lemma 3.6. Let M is convex set and $\phi(x)$ convex on M , $\psi(x)$ concave on M , then the efficient solution $\bar{x}$ for P_2 and P_3 is efficient solution for P_1 .

Proof.

Let $\bar{x}$ be an efficient solution for P_2 and P_3 and not efficient solution for P_1, then there exist $x^{**} \in M$ such that

$$\frac{\phi(x^{**})}{\psi(x^{**})} \le \frac{\phi(\bar{x})}{\psi(\bar{x})},\ \ \frac{\phi(x^{**})}{\psi(x^{**})} \le \frac{\phi(\bar{x})}{\psi(\bar{x})}$$

Since $\bar{x}$ is an efficient solution for P_2 and P_3 , then

$$\phi(\bar{x}) \le \phi(x^{**}) \qquad (1)$$

$$\psi(x^{**}) \le \psi(\bar{x})$$

$$\frac{1}{\psi(\bar{x})} \le \frac{1}{\psi(x^{**})} \qquad (2)$$

Multiple (1) on $\psi(x^{**})$,we get

$$\phi(\bar{x}).\psi(x^{**}) \le \phi(x^{**})\psi(x^{**})$$

$$\frac{\phi(\bar{x})}{\psi(x^{**})} \le \frac{\phi(x^{**})}{\psi(x^{**})} \qquad (3)$$

From (2) on (3), we get

$$\frac{\phi(\bar{x})}{\psi(\bar{x})} \le \frac{\phi(\bar{x})}{\psi(x^{**})} \le \frac{\phi(x^{**})}{\psi(x^{**})}$$

So,

$$\frac{\phi(\bar{x})}{\psi(\bar{x})} \le \frac{\phi(x^{**})}{\psi(x^{**})}$$

Which contradict the assumption, hence $\bar{x}$ is efficient solution for P_1.

Lemma 3.7. Let M is convex set and $\phi(x)$ convex on M , then if x^* is efficient solution for P_1, then $\bar{x}$ and $\bar{\bar{x}}$ are efficient solutions for P_2 and P_3 respectively .

Proof.

Since $x^* \in M^*$, $M_1^* \le M^* \le M_2^*$, then $\phi(\bar{x}) \le \dfrac{\phi(x^*)}{\psi(x^*)} \le \psi(\bar{\bar{x}})$

Let $\bar{x}$ not efficient solution for P_2, then there exist $x^{**} \in M$ such that $\phi(x^{**}) \le \phi(\bar{x})$, $\phi(x^{**}) \ne \phi(\bar{x})$.

Since M is a convex set, then from convexity there exist $\hat{x} \in M$, such that

$$\hat{x} = (1-\lambda)\ \bar{x} + \lambda\ x^{**},\ 0 \le \lambda \le 1$$

Let $\phi(x)$ is a convex set, then from convexity, we get

$$\phi(\hat{x}) \le (1-\lambda)\ \phi(\bar{x}) + \lambda\ \phi(x^{**}),$$

at $\lambda = 0$, we get

$$\phi(\hat{x}) \le \phi(\bar{x})$$

Which contradict the assumption, hence $\bar{x}$ efficient solution for P_2.

Let $\bar{\bar{x}}$ not efficient solution for P_3, then there exist $x^{**} \in M$ such that $\psi(x^{**}) \ge \psi(\bar{\bar{x}})$, $\psi(x^{**}) \ne \psi(\bar{\bar{x}})$.

Since M is a convex set, then from convexity there exist $\hat{x} \in M$, such that

$$\hat{x} = (1-\lambda)\ \bar{\bar{x}} + \lambda\ x^{**},\ 0 \le \lambda \le 1$$

Let $\psi(x)$ is a convex set , then from convexity ,we get

$$\psi(\hat{x}) \ge (1-\lambda)\ \psi(\bar{\bar{x}}) + \lambda\ \psi(x^{**}),$$

at $\lambda = 0$, we get

$$\psi(\hat{x}) \ge \psi(\bar{\bar{x}})$$

Which contradict the assumption, hence $\bar{\bar{x}}$ efficient solution for P_3.

4. The algorithm

From the previous discussion, the proposed algorithm proceeds as follows

Step1:

Construct two auxiliary problems P_2 and P_3 from P_1 as following:

$$P_2 \begin{cases} Min\ \phi(x) \\ s.t \\ M = \left\{X \in R^n : g_r(x) \le 0, r = 1,2,...,m\right\} \end{cases},\quad P_3 \begin{cases} Max\ \psi(x) \\ s.t \\ M = \left\{X \in R^n : g_r(x) \le 0, r = 1,2,...,m\right\} \end{cases}$$

Step2:

Solve the problems P_2 and P_3, get the set of all efficient solutions M_1^* and M_2^* for P_2 and P_3 respectively

$$M_1^* = \{\bar{x} : \bar{x} \in M\},\ M_2^* = \{\bar{\bar{x}} : \bar{\bar{x}} \in M\}.$$

Step3:

If $M_1^* \subset M_2^*$, then the set of all efficient solution M^* for P_1 is $M^* = M_1^* \cap M_2^* = M_1^*$, and the efficient solution is $\bar{x}$. Otherwise go to step 4.

Step4:
If $M_2^* \subset M_1^*$, then the set of all efficient solution M^* for P_1 is $M^* = M_1^* \cup M_2^* = M_1^*$, and the efficient solution is $\bar{x}$.

Step5:
If $M_1^* \cap M_2^* = \phi$, then the set of all efficient solution M^* for P_1 is $M_1^* \le M^* \le M_2^*$, and the efficient solution is x^*.

5. Illustrative examples

Example 1

$$\underset{x\ge 1}{Min}\left\{\frac{x+1}{(x-2)^2}\ ,\ \frac{(x-3)^2+1}{x^2}\right\}$$

Solution Steps:
Step 1:

$$(P_2)\ \{\ \underset{x\ge 1}{Min}\{x+1,(x-3)^2+1\}\},\quad (P_3)\ \{\ \underset{x\ge 1}{Max}\{(x-2)^2,x^2\}\}$$

Step 2:

$$M_1^* = \{\bar{x} : 1 \le \bar{x} \le 3\},\quad M_2^* = \{\bar{\bar{x}} : 1 \le \bar{\bar{x}} \le 2\}$$

Step 3:
Since $M_2^* \subset M_1^*$, then the set of all efficient solution M^* for P_1 is $M^* = M_1^* \cup M_2^* = M_1^*$, then $M^* = \{\bar{x} : 1 \le \bar{x} \le 3\}$

Example 2

$$\underset{x\ge 1}{Min}\left\{\frac{x+1}{-x}\ ,\ \frac{(x-3)^2+1}{2x}\right\}$$

Solution Steps:
Step 1:

$$(P_2)\ \{\ \underset{x\ge 1}{Min}\{x+1,(x-3)^2+1\}\},\quad (P_3)\ \{\ \underset{x\ge 1}{Max}\{-x,2x\}\}$$

Step 2:

$$M_1^* = \{\bar{x} : 1 \le \bar{x} \le 3\},\quad M_2^* = \{\bar{\bar{x}} : \bar{\bar{x}} \ge 1\}$$

Step 3:
Since $M_1^* \subset M_2^*$, then the set of all efficient solution M^* for P_1 is $M^* = M_1^* \cap M_2^* = M_1^*$, then $M^* = \{\bar{x} : 1 \le \bar{x} \le 3\}$

Example 3

$$\underset{x\ge 1}{Min}\left\{\frac{x+1}{x^2}\ ,\ \frac{(x-3)^2+1}{2x}\right\}$$

Solution Steps:
Step 1:

$$(P_2)\ \{\ \underset{x\geq 1}{Min}\{x+1,(x-3)^2+1\}\},\quad (P_3)\ \{\ \underset{x\geq 1}{Max}\{x^2,2x\}\}$$

Step 2:

$$M_1^*=\{\bar{x}:1\leq \bar{x}\leq 3\},\quad M_2^*=\{\bar{\bar{x}}:\bar{\bar{x}}=2\}$$

Step 3:
Since $M_2^*\subset M_1^*$, then the set of all efficient solution M^* for P_1 is $M^*=M_1^*\cup M_2^*=M_1^*$, then $M^*=\{\bar{x}:1\leq \bar{x}\leq 3\}$

Example 4

$$\underset{x\geq 1}{Min}\left\{\frac{(x-2)^2}{x+1}\ ,\ \frac{x^2}{(x-3)^2+1}\right\}$$

Solution Steps:
Step 1:

$$(P_2)\ \{\ \underset{x\geq 1}{Min}\{(x-2)^2,x^2\}\},\quad (P_3)\ \{\ \underset{x\geq 1}{Max}\{x+1,(x-3)^2+1\}\}$$

Step 2:

$$M_1^*=\{\bar{x}:1\leq \bar{x}\leq 2\},\quad M_2^*=\{\bar{\bar{x}}:1\leq \bar{\bar{x}}\leq 3\}$$

Step 3:
Since $M_1^*\subset M_2^*$, then the set of all efficient solution M^* for P_1 is $M^*=M_1^*\cap M_2^*=M_1^*$, then $M^*=\{\bar{x}:1\leq \bar{x}\leq 2\}$

Example 5

$$\underset{1\leq x\leq 5}{Min}\left\{\frac{1}{x}\ ,\ \mathrm{x}\right\}$$

Solution Steps:
Step 1:

$$(P_2)\ \{\ \underset{1\leq x\leq 5}{Min}\{1,x\}\},\quad (P_3)\ \{\ \underset{1\leq x\leq 5}{Max}\{x,1\}\}$$

Step 2:

$$M_1^*=\{\bar{x}:\bar{x}=1\},\quad M_2^*=\{\bar{\bar{x}}:\bar{\bar{x}}=5\}$$

Step 3:
Since $M_1^*\cap M_2^*=\phi$, then the set of all efficient solution M^* for P_1 is $M_1^*\leq M^*\leq M_2^*$, then $M^*=\{\bar{x}:1\leq x^*\leq 5\}$

6. Conclusion

In this paper, a new approach for finding all efficient solutions for multi-objective fractional programming problems is presented. This approach based on solving auxiliary problems in which minimize the numerator and maximize the denominator. Illustrative examples are presented to clarify the obtained results.

7. References

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