Concomitants of Order Statistics and Record Values from General Farlie-Gumble-Morgenstern Type Bivariate-Generalized Exponential Distribution

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Abstract

We introduce the Bairamov-Kotz-Becki-Farlie-Gumble-Morgenstern (BKB-FGM) type bivariate-generalized exponential distribution. Some distributional properties of concomitants of order statistics as well as record values for this family are studied. Recurrence relations between the moments of concomitants are obtained, some of these recurrence relations were not publishes before for Morgenstern type bivariate distributions. Moreover, most of the paper results are extended to arbitrary distributions (see Remark 3.1).

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Key Words: Concomitants; Order statistics; Record values; Generalized exponential distribution; BKB-FGM family.

1. Introduction

The generalized exponential distribution (GE), a most attractive generalization of the exponential distribution, introduced by Gupta and Kundu (1999), has widespread interest and applications, e.g., it can be used quite effectively in analyzing many lifetime data, particularly in place of two-parameter gamma and two-parameter Weibull distributions. The GE distribution has a nice physical interpretation. Suppose, there are $n-$components in a parallel system and the lifetime distribution of each component is independent and identically distributed. If the lifetime distribution of each component is GE, then the

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lifetime distribution of the system is also GE. As opposed to Weibull distribution, which represents a series system, GE represents a parallel system. Many authors studied various properties of the GE. See, for example, AL-Hussaini and Ahsanullah (2015), Ahsanullah et al. (2013), Kundu and Pradham (2009), among others. Nadarajah (2011) surveyed the GE distribution.

A continuous random variable (rv) is said to have the GE with scale parameter $\theta > 0$ and shape parameter $\alpha > 0$ (denoted by $GE(\theta;\alpha)$), if the probability density function (pdf) and the corresponding cumulative distribution function (cdf) are given, for $x > 0$, respectively, by

$$f_X(x) = \alpha\theta(1 - e^{-\theta x})^{\alpha-1}e^{-\theta x} \tag{1.1}$$

and

$$F_X(x) = (1 - e^{-\theta x})^{\alpha}. \tag{1.2}$$

Its hazard function is

$$H_X(x) = \frac{\alpha\theta(1 - e^{-\theta x})^{\alpha-1}e^{-\theta x}}{1 - (1 - e^{-\theta x})^{\alpha}}.$$

Gupta and Kundu (1999) showed that the kth moment of $GE(\theta;\alpha)$ is

$$\mu_k(\theta,\alpha) = \frac{\alpha k!}{\theta^k}\sum_{i=0}^{\aleph(\alpha-1)}\frac{(-1)^i}{(i+1)^{k+1}}\binom{\alpha-1}{i},$$

where $\aleph(x) = \infty$, if x is non-integer and $\aleph(x) = x$, if x is integer. Moreover, the mean, variance and moment generating function of $GE(\theta;\alpha)$ are given, respectively, by

$$\mu_1(\theta,\alpha) = \mathrm{E}(X) = \frac{B(\alpha)}{\theta}, \quad \mathrm{Var}(X) = \frac{C(\alpha)}{\theta^2} \ \text{ and } M_X(t) = \alpha\beta(\alpha, 1 - \frac{t}{\theta}), \tag{1.3}$$

where $B(\alpha) = \Psi(\alpha+1) - \Psi(1)$, $C(\alpha) = \Psi'(1) - \Psi'(\alpha+1)$, $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $\Psi(.)$ is the digamma function, while $\Psi'(.)$ is its derivation (the trigamma function). Moreover, the higher central moments can be obtained in terms of the polygamma functions.

In this paper we introduce the BKB-FGM type bivariate-generalized exponential distribution. Bairamov et al. (2001) presented a four-parameter extension of the classical Farlie-Gumble-Morgenstern (FGM) family of distributions, which allow to increase the dependence between the variables. Bairamov et al. (2001) considered a most general form of FGM model, where the cdf and pdf of this extension are given by

$$F(x,y) = F_X F_Y[1 + \lambda(1 - F_X^{p_1})^{q_1}(1 - F_Y^{p_2})^{q_2}], p_1, p_1, q_1, q_2 \geq 1, \tag{1.4}$$

$$f(x,y) = f(x)f(y)[1 + \lambda(1 - F_X^{p_1})^{q_1-1}(1 - (1 + p_1q_1)F_X^{p_1})(1 - F_Y^{p_2})^{q_2-1}(1 - (1 + p_2q_2)F_Y^{p_2})], \tag{1.5}$$

where $F_X(x)$ and $F_Y(y)$ are cdf's, while $f_X(x)$ and $f_Y(y)$ are pdf's of the rv's X and Y, respectively. The admissible range of the associated parameter λ is

$$\underline{\lambda}(p_1,p_1;p_2,q_2) = -\min\{1, \frac{1}{l_1(p_1,q_1)l_2(p_2,q_2)}\} \leq \lambda$$

$$\leq \min\left\{\frac{1}{l_1(p_1,q_1)}, \frac{1}{l_2(p_2,q_2)}\right\} = \overline{\lambda}(p_1,p_1;p_2,q_2), \tag{1.6}$$

where,

$$l_i(p_i,q_i) = \begin{cases} p_i(\frac{p_i(q_i-1)}{1+p_iq_i})^{q_i-1} & q_i > 1, \\ \frac{1}{p_i} & q_i = 1, i = 1,2. \end{cases}$$

The HK-FGM model (denoted by HK-FGM(λ,p)) suggested by Huang and Kotz (1999) and the classical FGM family are obtained as special cases of the BKB-FGM family (1.4) by choosing $p_i = p, q_i = 1, i = 1,2$ and $p_i = q_i = 1, i = 1,2$, respectively.

In this paper, all the results of Tahmasebi and Jafari (2015) are extended to BKB-FGM family with two marginals F_X and F_Y, where $X \sim GE(\theta_1;\alpha_1)$ and $Y \sim GE(\theta_2;\alpha_2)$ (denoted by BKB-FGM-GE$(\theta_1,\alpha_1;\theta_2,\alpha_2)$). Moreover, some new results, which were not obtained by Tahmasebi and Jafari (2015) for FGM family, are given such as recurrence relations between the moments of concomitants of order statistics. It is worth mentioning that most of the obtained recurrence relations are valued for any arbitrary distributions.

2. Some properties of BKB-FGM-GE$(\theta_1,\alpha_1;\theta_2,\alpha_2)$ and the motivation of the work

In this section we study the correlation coefficient of the model BKB-FGM-GE$(\theta_1,\alpha_1;\theta_2,\alpha_2)$ and we show that it is large than the correlation coefficient of the two models HK-FGM-GE$(\theta_1,\alpha_1;\theta_2,\alpha_2)$ and FGM-GE $(\theta_1, \alpha_1;\theta_2,\alpha_2)$. Now, by using the Hoeffding formula (see Lehmann, 1966), we get

$$\mathrm{COV}(X,Y:\lambda;\alpha_1,\alpha_2;p_1,p_2;q_1,q_2) = \int_0^\infty \int_0^\infty [F(x,y) - F(x)F(y)]dxdy$$

$$= \int_0^\infty \int_0^\infty \lambda F(x)F(y)(1-F_X^{p_1})^{q_1}(1-F_Y^{p_2})^{q_2}dxdy = \frac{\lambda}{\theta_1\theta_2}I_1I_2, \tag{2.1}$$

where

$$I_i = \int_0^1 \xi^{\alpha_i}(1-\xi^{\alpha_i p_i})^{q_i}\frac{1}{1-\xi}d\xi = \sum_{j=0}^\infty \int_0^1 \xi^{\alpha_i+j}(1-\xi^{\alpha_i p_i})^{q_i}d\xi$$

$$= \frac{1}{\alpha_i p_i}\sum_{j=0}^\infty \int_0^1 \xi^{\frac{\alpha_i+j+1}{\alpha_i p_i}-1}(1-\xi)^{q_i}d\xi = \frac{1}{\alpha_i p_i}\sum_{j=1}^\infty \beta(\frac{\alpha_i+j+1}{\alpha_i p_i}, q_i+1). \tag{2.2}$$

Therefore, the correlation coefficient of the BKB-FGM-GE$(\theta_1, \alpha_1; \theta_2, \alpha_2))$ is given by

$$\rho(X, Y : \lambda; \alpha_1, \alpha_2; p_1, p_2; q_1, q_2) = \lambda \prod_{i=1}^{2} \frac{1}{\sqrt{C(\alpha_i)}\alpha_i p_i} \sum_{j=1}^{\infty} \beta(\frac{\alpha_i + j + 1}{\alpha_i p_i}, q_i + 1). \tag{2.3}$$

Remark 2.1. For $p_1 = p_2 = p$ and $q_1 = q_2 = 1$, we get

$$\mathrm{COV}(X, Y : \lambda; \alpha_1, \alpha_2; p_1, p_2; q_1, q_2)$$

$$= \frac{\lambda}{p^2} \prod_{i=1}^{2} \frac{1}{\theta_i \alpha_i} \sum_{j=0}^{\infty} \frac{\Gamma(\frac{\alpha_i+j+1}{\alpha_i p})}{\Gamma(\frac{\alpha_i+j+1}{\alpha_i p} + 2)} = \frac{\lambda}{p^2} \prod_{i=1}^{2} \frac{1}{\theta_i \alpha_i} \sum_{j=0}^{\infty} \frac{1}{(\frac{\alpha_i+j+1}{\alpha_i p} + 1)(\frac{\alpha_i+j+1}{\alpha_i p})}$$

$$= \frac{\lambda}{\theta_1 \theta_2} \prod_{i=1}^{2} \sum_{j=0}^{\infty} \left[\frac{1}{\alpha_i + j + 1} - \frac{1}{\alpha_i(1+p) + j + 1} \right] = \frac{\lambda}{\theta_1 \theta_2} \prod_{i=1}^{2} (\Psi(\alpha_i(1+p)+1) - \Psi(\alpha_i + 1)).$$

Therefore, for the family HK-FGM-GE$(\theta_1, \alpha_1; \theta_2, \alpha_2)$ we get

$$\rho(X, Y : \lambda; \alpha_1, \alpha_2; p, p; 1, 1) = \lambda \frac{D(\alpha_1, p) D(\alpha_2, p)}{\sqrt{C(\alpha_1(p+1)) C(\alpha_2(p+1))}},$$

where $D(\alpha_i, p) = B(\alpha_i(1+p)) - B(\alpha_i)$. Barakat et al. (2017) showed that

$$\underline{\rho}(p) = -\frac{6(\log(p+1))^2}{\pi^2 p^2} \leq \lim_{\substack{\alpha_1 \to 0 \\ \alpha_2 \to 0}} \rho(X, Y : \lambda; \alpha_1, \alpha_2; p, p; 1, 1)$$

$$= \rho(X, Y : \lambda; p) \leq \frac{6(\log(p+1))^2}{\pi^2 p} = \overline{\rho}(p),$$

which yields $\rho(X, Y : \lambda; p) \to 0$, as $p \to \infty$, and $|\rho(X, Y : \lambda; 1)| \leq 0.2921$ (i.e., when $p = 1$). Moreover, $\overline{\rho}(3.9241) \leq 0.3937$, which is a significant improvement comparing with the upper bound "0.2921"obtained by Tahmasebi and Jafari (2015).

The following theorem gives some interesting properties of the family BKB-FGM-GE$(\theta_1, \alpha_1; \theta_2, \alpha_2)$.

Theorem 2.1. Let $\underline{\lambda}(p_1, p_2; 1, 1) \leq \lambda \leq \overline{\lambda}(p_1, p_2; 1, 1)$. Then,

$$\rho(X, Y : \lambda; \alpha_1, \alpha_2; p_1, p_2; q_1, q_2) \leq \rho(X, Y : \lambda; \alpha_1, \alpha_2; p_1, p_2; 1, 1), \forall q_1, q_2 > 1. \tag{2.4}$$

Moreover,

$$\lim_{\substack{\alpha_1 \to 0 \\ \alpha_2 \to 0}} \rho(X, Y : \lambda; \alpha_1, \alpha_2; p_1, p_2; q_1, q_2) = 0,$$

and

$$\lim_{\substack{\alpha_1 \to \infty \\ \alpha_2 \to \infty}} \rho(X, Y : \lambda; \alpha_1, \alpha_2; p_1, p_2; q_1, q_2) = \rho(X, Y : \lambda; p_1, p_2; q_1, q_2)$$

$$= \frac{6\lambda}{\pi^2} \prod_{i=1}^{2} \frac{1}{\theta_i} \int_0^1 (1 - z^{p_i})^{q_i} \frac{1}{\log z} dz. \tag{2.5}$$

Finally, if q_1 and q_2 are integers, we get

$$\rho(X,Y:\lambda;p_1,p_2;q_1,q_2)=\frac{6\lambda}{\pi^2}\prod_{i=1}^{2}\frac{1}{\theta_i}\sum_{j=1}^{q_i}(-1)^j\binom{q_i}{j}\log(1+p_ij). \tag{2.6}$$

Remark 2.2. The relation (2.4) reveals an interesting fact that whenever λ is belonging to the admissible range of HK-FGM model, the correlation coefficient of the HK-FGM model is greater than the BKB-FGM model. However, we will show that outside this admissible range, the correlation coefficient of the BKB-FGM model may be extremely greater than the correlation coefficient of the HK-FGM model.

Remark 2.3. At $q_1=q_2=1$, $p_1=p_2=p$, the R.H.S. of (2.6), becomes $\frac{\lambda}{\theta_1\theta_2}\log^2(1+p)$, which coincides with the result of Barakat et al. (2017).

Proof of Theorem 2.1. The proof of the relation (2.4) follows immediately from the fact that the function $\beta(x,y)$ is non-increasing in both x and y, and $q_i\geq 1$. In order to prove the relation (2.5), we start with the relation (2.1) with

$$I_i=\int_0^1\xi^{\alpha_i}(1-\xi^{\alpha_ip_i})^{q_i}\frac{1}{1-\xi}d\xi=\frac{1}{\alpha_i}\int_0^1(1-z^{p_i})^{q_i}\frac{z^{\frac{1}{\alpha_i}}}{1-z^{\frac{1}{\alpha_i}}}dz \tag{2.7}$$

(by using the transformation $z=\xi^{\alpha_i}$). On the other hand, for any $0<z<1$, we have

$$\lim_{\alpha_i\to\infty}\frac{1}{\alpha_i(1-z^{\frac{1}{\alpha_i}})}=\lim_{\theta\to0}\frac{\theta}{1-z^{\theta}}=\lim_{\theta\to0}\frac{1}{z^{\theta}\log z}=-\frac{1}{\log z} \tag{2.8}$$

and

$$\lim_{\alpha_i\to\infty}C(\alpha_i)=\frac{\pi^2}{6}. \tag{2.9}$$

Combining (2.7), (2.8) and (2.9), we get the relation (2.5). Finally, the relation (2.8) follows immediately by using the relation $\int_0^1(z^{p_i}-1)^{q_i}\frac{1}{\log z}dz=\sum_{j=1}^{q_i}(-1)^{q_i-j}\binom{q_i}{j}\log(1+p_ij)$ (cf. Prudnikov et al., 1998). $\square$

Corollary 2.1. Simple calculations reveal that, when $q_1=q_2=q$ and $p_1=p_2=p$, where both of q and p are integers, the correlation coefficient $\rho(X,Y:\lambda;p,p;q,q)$ attains its maximum "0.393602"at $q=2,p=8$. On the other hand, when $1<q<1.9$, the correlation coefficient $\rho(X,Y:\lambda;p,p;q,q)$ nearly attains its maximum "0.5518971"at $q=1.3,p=6$. This result represents a significant improvement comparing with the maximum value of the correlation coefficient in HK-FGM-GE family (which is 0.3937, cf. Barakat et al., 2017). Consequently, this fact gives a satisfactory motivation to deal with BKB-FGM-GE rather than HK-FGM-GE.

3. Concomitants of order statistics

Let $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ be a random sample from a bivariate cdf $F_{X,Y}(x, y)$. If we arrange the $X-$variate in ascending order as $X_{1:n} \leq X_{2:n} \leq .... \leq X_{n:n}$, then, the $Y-$variate paired with these order statistics are denoted by $Y_{[1:n]}, Y_{[2:n]}, ..., Y_{[n:n]}$ and termed the concomitants of order statistics. The concept of concomitants of order statistics was first introduced by David (1973) and almost simultaneously under the name of induced order statistics by Bhattacharya (1974). These concomitant order statistics are of interest in selection and prediction problems based on the ranks of the X's. Another application of concomitants of order statistics is in ranked-set sampling. It is a sampling scheme for situations where measurement of the variable of primary interest for sampled items is expensive or time-consuming while ranking of a set of items related to the variable of interest can be easily done. A comprehensive review of ranked-set sampling can be found in Chen et al. (2004). For a recent comprehensive review of possible applications of the concomitants of order statistics, see David and Nagaraja (1998).

Let $X \sim GE(\theta_1; \alpha_1)$ and $Y \sim GE(\theta_2; \alpha_2)$. Since the conditional pdf of $Y_{[r:n]}$ given $X_{[r:n]} = x$ is $f_{Y_{[r:n]}|X_{r:n}}(y|x) = f_{Y|X}(y|x)$, then the pdf of $Y_{[r:n]}$ is given by

$$f_{[r:n]}(y) = \int_0^\infty f_{Y|X}(y|x) f_{r:n}(x) dx, \tag{3.1}$$

where $f_{r:n}(x) = \frac{1}{\beta(r,n-r+1)} F_X^{r-1}(1 - F_X(x))^{n-r} f_X(x)$ is the pdf of the rth order statistic $X_{r:n}$ and $f_{Y|X}(y|x)$ can be computed by using (1.1), (1.2) and (1.5). The following simple proved theorem gives the useful representation of the pdf $f_{[r:n]}(y)$.

Theorem 3.1. Let $U_i \sim GE(\theta_2, \alpha_2(ip_2 + 1))$ and $V_i \sim GE(\theta_2, \alpha_2((i + 1)p_2 + 1))$. Then

$$f_{[r:n]}(y) = f_Y(y) + \mathcal{S}_{r,n}^{(t)}(p_1, q_1)$$

$$\times \sum_{i=0}^{\aleph(q_2-1)} \binom{q_2 - 1}{i} (-1)^i \left\{ \frac{1}{ip_2 + 1} f_{U_i}(y) - \frac{1 + p_2 q_2}{(i + 1)p_2 + 1} f_{V_i}(y) \right\}, t = 1, 2,$$

where

$$\mathcal{S}_{r,n}^{(1)}(p_1, q_1) = \frac{\lambda}{p_1} \sum_{j=0}^{n-r} \binom{n - r}{j} (-1)^j \Delta^\star_{j:r,n}(p_1, q_1),$$

$$\Delta^\star_{j:r,n}(p_1, q_1) = \frac{\beta(\frac{r+j}{p_1}, q_1) - (1 + p_1 q_1)\beta(\frac{r+j}{p_1} + 1, q_1)}{\beta(r, n - r + 1)},$$

$$\mathcal{S}_{r,n}^{(2)}(p_1, q_1) = \lambda \sum_{j=0}^{\aleph(q_1-1)} \binom{q_1 - 1}{j} (-1)^j \Delta^{\star\star}_{j:r,n}(p_1, q_1)$$

and

$$\Delta^{\star\star}_{j:r,n}(p_1,q_1)=\frac{\beta(jp_1+r,n-r+1)-(1+p_1q_1)\beta((j+1)p_1+r,n-r+1)}{\beta(r,n-r+1)}.$$

Proof. Clearly, the relation (3.1), can be written in the form

$$f_{[r:n]}(y)=f_Y(y)\left[1+\lambda\left(1-F_Y^{p_2}(y)\right)^{q_2-1}\left(1-(1+p_2q_2)F_Y^{p_2}(y)\right)(J_1-(1+p_1q_1)J_2)\right]$$

$$=f_Y(y)+(J_1-(1+p_1q_1)J_2)\sum_{i=0}^{\aleph(q_2-1)}\binom{q_2-1}{i}(-1)^i\left\{\frac{1}{ip_2+1}f_{U_i}(y)-\frac{1+p_2q_2}{(i+1)p_2+1}f_{V_i}(y)\right\},$$

where

$$J_1\beta(r,n-r+1)=\int_0^\infty\left(1-F_Y^{p_1}(x)\right)^{q_1-1}F_X^{r-1}(x)(1-F_X(x))^{n-r}f_X(x)dx$$

and

$$J_2\beta(r,n-r+1)=\int_0^\infty\left(1-F_Y^{p_1}(x)\right)^{q_1-1}F_X^{p_1+r-1}(x)(1-F_X(x))^{n-r}f_X(x)dx.$$

Now, the proof of the relations $\mathcal{S}^{(1)}_{r,n}(p_1,q_1)=J_1-(1+p_1q_1)J_2$ and $\mathcal{S}^{(2)}_{r,n}(p_1,q_1)=J_1-(1+p_1q_1)J_2$ follows immediately, upon using the transformations $Z=F_X^{p_1}(x)$ and $W=F_X(x)$ and then applying the binomial theorem on the resulted terms $(1-Z^{\frac{1}{p_1}})^{n-r}$ and $(1-W^{p_1})^{q_1-r}$ in $J_1-(1+p_1q_1)J_2$, respectively. □

The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.1. Let $\mu^{(k)}_{[r:n]}=\mathrm{E}(Y^k_{[r:n]}),\ k\in\Re^+$. Then,

$$\mu^{(k)}_{[r:n]}=\mu_k(\theta_2,\alpha_2)+\mathcal{S}^{(t)}_{r,n}(p_1,q_1)\mathcal{D}(k;p_2,q_2)$$

$$=\mu_k(\theta_2,\alpha_2)+\mathcal{S}^{(t)}_{r,n}(p_1,q_1)\sum_{i=0}^{\aleph(q_2-1)}\binom{q_2-1}{i}(-1)^i\left\{\frac{\mathrm{E}(U_i^k)}{ip_2+1}-\frac{(1+p_2q_2)\mathrm{E}(V_i^k)}{(i+1)p_2+1}\right\},t=1,2, \tag{3.2}$$

where, $\mathrm{E}(U_i^k)$ and $\mathrm{E}(V_i^k)$, can be easily computed by using the relation (1.3). Therefore, the mean $\mu_{[r:n]}=\mathrm{E}(Y_{[r:n]})$ is given by

$$\mu_{[r:n]}=\frac{B(\alpha_2)}{\theta_2}+\mathcal{S}^{(t)}_{r,n}(p_1,q_1)\mathcal{D}(1,p_2,q_2),t=1,2, \tag{3.3}$$

where

$$\mathcal{D}(1,p_2,q_2)=\frac{1}{\theta_2}\sum_{i=0}^{\aleph(q_2-1)}\binom{q_2-1}{i}(-1)^i\left[\frac{B(\alpha_2(ip_2+1))}{ip_2+1}-\frac{(1+p_2q_2)B(\alpha_2((i+1)p_2+1))}{(i+1)p_2+1}\right].$$

Corollary 3.2. When $q_1=q_2=1$ and $p_1=p_2=p$, i.e., for the family HK-FGM-GE, we get

$$\mu_{[r:n]}=\frac{1}{\theta_2}\left[(1+\lambda\Delta^{\star\star}_{0:r,n}(p,1))B(\alpha_2)-\lambda\Delta^{\star\star}_{0:r,n}(p,1)B(\alpha_2(p+1))\right]. \tag{3.4}$$

Proof of Corollary 3.2. Clearly $\mathcal{D}(1,p,1)=\frac{1}{\theta_2}[B(\alpha_2)-B(\alpha_2(p+1))]$ and $\mathcal{S}^{(2)}_{r,n}(p,1)=\lambda\Delta^{\star\star}_{0:r,n}(p,1)$. Therefore, the proof of (3.4) is obvious for $t=2$. On the other hand, upon using the relation (cf. Kamps, 1995, Page 186)

$$\sum_{j=0}^{M}\binom{M}{j}(-1)^j\frac{1}{aj+b}=a^M M!\left[\prod_{t=0}^{M}(at+b)\right]^{-1},\forall aj+b\neq 0, M\in\Re^+, \qquad (3.5)$$

it is easy to show that

$$\mathcal{S}^{(1)}_{r,n}(p,1)=\frac{\lambda}{\beta(r,n-r+1)}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\left[\frac{1}{r+j}-\frac{1+p}{r+j+p}\right]=\lambda\Delta^{\star\star}_{0:r,n}(p,1),$$

(by choosing $M=n-r, a=1, b=r$ and $M=n-r, a=1, b=r+p$ in the first and the second term of the last summation, respectively). $\square$

Remark 3.1. It is worth mentioning that, if we replace $\mu_k(\theta_2,\alpha_2)$ by $\mathrm{E}(Y^k)$. Moreover, U_i and V_i in $\mathcal{D}(k;p_2,q_2)$ are taken to be such that $U_i\sim F_X^{ip_2+1}$ and $V_i\sim F_Y^{(i+1)p_2+1}$, then the representation (3.2) holds for any two arbitrary distributions F_X and F_Y.

Now, by using the two representations in relation (3.2), as well as (3.3), at $t=1$ and $t=2$, we can derive some useful recurrence relations satisfied by the moments $\mu^{(k)}_{[r:n]}, k=1,2,...$. The following theorem give a new recurrence relation by using the representation at $t=1$. It is worth mentioning that this recurrence relation was not proved even for the model FGM-GE. Moreover, in view of Remark 3.1, all the next recurrence relations are satisfied for arbitrary distributions F_X and F_Y, if only we would consider the obvious changes illustrated in Remark 3.1.

Theorem 3.2. Let p_1 be an integer, then

$$\frac{1}{m^{(2)}_{r,n}(p_1)}\mu^{(k)}_{[r+2p_1:n+2p_1]}+\frac{1}{m^{(1)}_{r,n}(p_1)}\mu^{(k)}_{[r+p_1:n+p_1]}-2\mu^{(k)}_{[r:n]}=$$

$$\left(\frac{1}{m^{(2)}_{r,n}(p_1)}+\frac{1}{m^{(1)}_{r,n}(p_1)}-2\right)\mu_k(\theta_2,\alpha_2)-\frac{\lambda}{p_1}\mathcal{D}(k,p_2,q_2)\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\eta_{j:r,n}(p_1,q_1),$$

where

$$m^{(i)}_{r,n}(p_1)=\frac{\Gamma(r)\Gamma(n+ip_1+1)}{\Gamma(r+ip_1)\Gamma(n+1)}, i=1,2,$$

and

$$\eta_{j:r,n}(p_1,q_1)=\left(3-\frac{p_1(1+q_1)}{r+p_1(1+q_1)+j}\right)\frac{\beta(\frac{r+j}{p_1},q_1+1)}{\beta(r,n-r+1)}$$
$$-(1+p_1q_1)\left(3-\frac{p_1(1+q_1)}{r+p_1(2+q_1)+j}\right)\frac{\beta(\frac{r+j}{p_1}+1,q_1+1)}{\beta(r,n-r+1)}.$$

Proof. Starting with $\Delta^{\star}_{j:r,n}(p_1,q_1)$, after simple calculations, we can show that

$$\Delta^{\star}_{j:r+p_1,n+p_1}(p_1,q_1)=m^{(1)}_{r,n}(p_1)[\Delta^{\star}_{j:r,n}(p_1,q_1)-\xi_1],$$

where

$$\xi_1 = \frac{\beta(\frac{r+j}{p_1}, q_1+1) - (1+p_1q_1)\beta(\frac{r+j}{p_1}+1, q_1+1)}{\beta(r, n-r+1)}.$$

Therefore,

$$\frac{1}{m^{(1)}_{r,n}(p_1)}\Delta^{\star}_{j:r+p_1,n+p_1}(p_1,q_1) - \Delta^{\star}_{j:r,n}(p_1,q_1) = -\xi_1. \tag{3.6}$$

Similarly, after some calculations, we get

$$\Delta^{\star}_{j:r+2p_1,n+2p_1}(p_1,q_1) = m^{(2)}_{r,n}(p_1)[\Delta^{\star}_{j:r,n}(p_1,q_1) - \xi_2], \tag{3.7}$$

where

$$\xi_2 = \xi_1 + \frac{\frac{j+r}{p_1}\beta(\frac{r+j}{p_1}, q_1+1)}{(\frac{r+j}{p_1}+q_1+1)\beta(r,n-r+1)} - \frac{(1+p_1q_1)(1+\frac{j+r}{p_1})\beta(\frac{r+j}{p_1}+1, q_1+1)}{(\frac{j+r}{p_1}+q_1+2)\beta(r,n-r+1)}.$$

Therefore,

$$\frac{1}{m^{(2)}_{r,n}(p_1)}\Delta^{\star}_{j:r+2p_1,n+2p_1}(p_1,q_1) - \Delta^{\star}_{j:r,n}(p_1,q_1) = -\xi_2. \tag{3.8}$$

Moreover, it is easy to show that $\eta_{j:r,n}(p_1,q_1) = \xi_1 + \xi_2$, thus by combining this equality with (3.6), (3.7), (3.8) and (3.2), at $t = 1$, the proof of the theorem follows immediately. □.

Corollary 3.3. For $q_1 = q_2 = 1$ and $p_1 = p_2 = p$, i.e., for HK-FGM-GE family, we get

$$\frac{1}{m^{(2)}_{r,n}(p)}\mu^{(k)}_{[r+2p:n+2p]} + \frac{1}{m^{(1)}_{r,n}(p)}\mu^{(k)}_{[r+p:n+p]} - 2\mu^{(k)}_{[r:n]} =$$

$$\left(\frac{1}{m^{(2)}_{r,n}(p)} + \frac{1}{m^{(1)}_{r,n}(p)} - 2\right)\mu_k(\theta_2,\alpha_2) - \frac{\lambda\mathcal{D}(k,p,1)}{p\beta(r,n-r+1)}\sum_{j=0}^{3} c_j(p)\beta(r+jp, n-r+1), \tag{3.9}$$

where $c_0(p) = 2p$, $c_1(p) = -p(2p+3)$, $c_2(p) = p^2$, $c_3(p) = p(1+p)$ and $\mathcal{D}(k,p,1) = \mu_k(\theta_2,\alpha_2) - \mu_k(\theta_2,\alpha_2(1+p))$. Moreover, For $q_1 = q_2 = 1$ and $p_1 = p_2 = 1$, i.e., for FGM-GE family, we get

$$\frac{r}{n+1}\mu^{(k)}_{[r+2:n+2]} + \frac{r+1}{n+2}\mu^{(k)}_{[r+1:n+1]} - 2\mu^{(k)}_{[r:n]} = -\frac{(n-r+1)(2n+3)}{(n+2)(n+1)}\mu_k(\theta_2,\alpha_2)$$
$$-\lambda\mathcal{D}(k,1,1)\left[2 - \frac{5r}{n+1} + \frac{(r+1)r}{(n+2)(n+1)} + \frac{2(r+2)(r+1)r}{(n+3)(n+2)(n+1)}\right]. \tag{3.10}$$

Proof. The proof of the relation $\mathcal{D}(k,p,1) = \mu_k(\theta_2,\alpha_2) - \mu_k(\theta_2,\alpha_2(p+1))$ is clear. On the other hand, after simple calculations, it can be shown that

$$\eta_{j:r,j}(p,1) = \frac{p^3}{\beta(r,n-r+1)}$$
$$\times\left[3\left(\frac{2-r-j}{(r+j+2p)(r+j+p)(r+j)}\right) - 2p\left(\frac{3-r-j}{(r+j+3p)(r+j+2p)(r+j+p)(r+j)}\right)\right]$$

$$= \frac{1}{\beta(r,n-r+1)}\left[\frac{p(1+p)}{j+(r+3p)}+\frac{p^2}{j+(r+2p)}-\frac{p(2p+3)}{j+(r+p)}+\frac{2p}{j+r)}\right].$$

Upon using the relation (3.5) and after simple calculations, we get the relation

$$\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\eta_{j:r,n}(p,1)=\frac{1}{\beta(r,n-r+1)}\sum_{j=0}^{3}c_j(p)\beta(r+jp,n-r+1).$$

This completes the proof of the relation (3.9). The proof of (3.10) is obvious. □

The following theorem, which is relying on the representation (3.2), at $t=2$, given some recurrence relations satisfied by the kth moments of concomitants of order statistics for any arbitrary distributions

Theorem 3.3. For any $k \in \Re^+$, we have

$$\frac{\mu^{(k)}_{[r+2:n]}-\mu^{(k)}_{[r:n]}}{\mu^{(k)}_{[r+1:n]}-\mu^{(k)}_{[r:n]}}=\frac{2r+1}{r+1}+\frac{p_1}{r+1}\Omega_{r,n}(p_1,q_1) \tag{3.11}$$

and

$$\frac{\mu^{(k)}_{[r:n-2]}-\mu^{(k)}_{[r:n]}}{\mu^{(k)}_{[r:n-1]}-\mu^{(k)}_{[r:n]}}=\frac{2n-1}{n-1}+\frac{p_1}{n-1}\Omega_{r,n}(p_1,q_1), \tag{3.12}$$

where

$$\Omega_{r,n}(p_1,q_1)$$

$$=\frac{\sum_{j=0}^{\aleph(q_1-1)}\binom{q_1-1}{j}(-1)^j\left[j^2\beta(jp_1+r,n-r+1)-(1+p_1q_1)(1+j)^2\beta((j+1)p_1+r,n-r+1)\right]}{\sum_{j=0}^{\aleph(q_1-1)}\binom{q_1-1}{j}(-1)^j\left[j\beta(jp_1+r,n-r+1)-(1+p_1q_1)(1+j)\beta((j+1)p_1+r,n-r+1)\right]}.$$

Proof. It is easy to check that

$$\mathcal{S}^{(2)}_{r+1,n}(p_1,q_1)-\mathcal{S}^{(2)}_{r,n}(p_1,q_1)=\frac{\lambda}{r\beta(r,n-r+1)}$$

$$\times\sum_{j=0}^{\aleph(q_1-1)}\binom{q_1-1}{j}(-1)^j\left[jp_1\beta(jp_1+r,n-r+1)-(1+p_1q_1)(1+j)p_1\beta((j+1)p_1+r,n-r+1)\right] \tag{3.13}$$

and

$$\mathcal{S}^{(2)}_{r+2,n}(p_1,q_1)-\mathcal{S}^{(2)}_{r,n}(p_1,q_1)=\frac{\lambda}{r(r+1)\beta(r,n-r+1)}$$

$$\times[\sum_{j=0}^{\aleph(q_1-1)}\binom{q_1-1}{j}(-1)^j[jp_1(jp_1+2r+1)\beta(jp_1+r,n-r+1)$$

$$-(1+p_1q_1)(1+j)p_1((j+1)p_1+2r+1)\beta((j+1)p_1+r,n-r+1)]. \tag{3.14}$$

Therefore, the proof of (3.11) follows after simple calculation, upon dividing (3.14) by (3.13) and using the representation (3.2). The proof of the relation (3.12) follows along the same way as the proof of (3.11), with only the obvious changes. □

Corollary 3.4. For $q_1 = q_2 = 1$ and $p_1 = p_2 = p$, i.e., for HK-FGM-GE family, we get

$$(r+1)\mu^{(k)}_{[r+2:n]} = (2r+p+1)\mu^{(k)}_{[r+1:n]} - (r+p)\mu^{(k)}_{[r:n]}$$

and

$$(n+p)\mu^{(k)}_{[r:n]} = (2n+p-1)\mu^{(k)}_{[r:n-1]} - (n-1)\mu^{(k)}_{[r:n-2]}.$$

Proof. The proof is obvious, since $\Omega_{r,n}(p,1) = 1$. □

Theorem 3.4. For any $k \in \Re^+$, we have

$$\mu^{(k)}_{[r+2:n]} + \mu^{(k)}_{[r+1:n]} - 2\mu^{(k)}_{[r:n]} = \mathcal{D}(k,p_2,q_2)\Omega^{(1)}_{r,n}(p_1,q_1),$$

where

$$\Omega^{(1)}_{r,n}(p_1,q_1) = \frac{\lambda p_1}{r(r+1)\beta(r,n-r+1)} \sum_{j=0}^{\aleph(q_1-1)} \binom{q_1-1}{j} (-1)^j [j(jp_1+3r+2)\beta(jp_1+r,n-r+1)$$
$$-(1+p_1q_1)(1+j)((j+1)p_1+3r+2)\beta((j+1)p_1+r,n-r+1)].$$

Moreover,

$$\mu^{(k)}_{[r:n-2]} + \mu^{(k)}_{[r:n-1]} - 2\mu^{(k)}_{[r:n]} = \mathcal{D}(k,p_2,q_2)\Omega^{(2)}_{r,n}(p_1,q_1),$$

where

$$\Omega^{(2)}_{r,n}(p_1,q_1) = \frac{\lambda p_1}{n(n-1)\beta(r,n-r+1)} \sum_{j=0}^{\aleph(q_1-1)} \binom{q_1-1}{j} (-1)^j [j(jp_1+3n-2)\beta(jp_1+r,n-r+1)$$
$$-(1+p_1q_1)(1+j)((j+1)p_1+3n-2)\beta((j+1)p_1+r,n-r+1)].$$

Proof. The proof of the theorem is similar to the proof of Theorem 3.3, with the exception that we carry out the addition operation instead of substraction operation (e.g. on the relations (3.13) and (3.14) for proving (3.11)). □

Corollary 3.5. For $q_1 = q_2 = 1$ and $p_1 = p_2 = p$, i.e., for HK-FGM-GE family, we get

$$\mu^{(k)}_{[r+2:n]} + \mu^{(k)}_{[r+1:n]} - 2\mu^{(k)}_{[r:n]} = -\frac{\lambda p(1+p)(p+3r+2)\beta(r+p,n-r+1)}{r(r+1)\beta(r,n-r+1)}\mathcal{D}(k,p,1).$$

When $p = 1$, i.e., for FGM-GE family, we get

$$\mu^{(k)}_{[r+2:n]} + \mu^{(k)}_{[r+1:n]} - 2\mu^{(k)}_{[r:n]} = -\frac{6\lambda}{n+1}\mathcal{D}(k,1,1).$$

Moreover, For $q_1 = q_2 = 1$ and $p_1 = p_2 = p$, i.e., for HK-FGM-GE family, we get

$$\mu^{(k)}_{[r:n-2]} + \mu^{(k)}_{[r:n-1]} - 2\mu^{(k)}_{[r:n]} = -\frac{\lambda p(1+p)(p+3n-2)\beta(r+p,n-r+1)}{n(n-1)\beta(r,n-r+1)}\mathcal{D}(k,p,1).$$

When $p = 1$, i.e., for FGM-GE family, we get

$$\mu^{(k)}_{[r:n-2]} + \mu^{(k)}_{[r:n-1]} - 2\mu^{(k)}_{[r:n]} = -\frac{2\lambda r(3n-1)}{(n-1)n(n+1)}\mathcal{D}(k,1,1).$$

Proof. The proof is obvious, since it follows after simple calculations. □

4. Concomitants of record values based on BKB-FGM-GE family

Let $\{(X_i, Y_i)\}, i = 1, 2, ...$ be a random sample from BKB-FGM-GE$(\theta_1, \alpha_1; \theta_2, \alpha_2)$. When the experimenter interests in studying just the sequence of records of the first component X_i's the second component associated with the record value of the first one is termed as the concomitant of that record value. The concomitants of record values has many applications, e.g., see Bdair and Raqab (2013) and Arnold et al. (1998). Some properties from concomitants of record values can be found in Ahsanullah (2009) and Ahsanullah and Shakil (2013). Let $\{R_n, n \geq 1\}$ be the sequence of record values in the sequence of X's while $R_{[n]}$ be the corresponding concomitant. Houchens (1984) obtained the pdf of concomitant of nth record value for $n \geq 1$, as $\mathcal{R}_{[n]}(y) = \int_0^\infty f_Y(y|x)h_n(x)dx$, where $h_n(x) = \frac{1}{\Gamma(n)}(-\log(1-F_X(x)))^{n-1}f_X(x)$ is the pdf of R_n. The following theorem gives a useful representation for the pdf $\mathcal{R}_{[n]}(y)$, as well as the kth moments concomitants of record values based on BKB-FGM-GE.

Theorem 4.1. Let U_i and V_i be defined as in Theorem 3.1. Then,

$$\mathcal{R}_{[n]}(y) = f_Y(y) + \lambda\left[\mathcal{S}^\star(p_1, q_1) - (1+p_1q_1)\mathcal{S}^{\star\star}(p_1, q_1)\right]$$

$$\times \sum_{i=0}^{\aleph(q_2-1)} \binom{q_2-1}{i}(-1)^i \left\{\frac{1}{ip_2+1}f_{U_i}(y) - \frac{1+p_2q_2}{(i+1)p_2+1}f_{V_i}(y)\right\}, \tag{4.1}$$

where

$$\mathcal{S}^\star(p_1, q_1) = \sum_{j=0}^{\aleph(q_1-1)}\sum_{\ell=0}^{\aleph(jp_1)}(-1)^{j+\ell}\frac{\binom{q_1-1}{j}\binom{jp_1}{\ell}}{(\ell+1)^n}$$

and

$$\mathcal{S}^{\star\star}(p_1, q_1) = \sum_{j=0}^{\aleph(q_1-1)}\sum_{\ell=0}^{\aleph(p_1(j+1))}(-1)^{j+\ell}\frac{\binom{q_1-1}{j}\binom{p_1(j+1)}{\ell}}{(\ell+1)^n}.$$

Moreover, if $\mu_{R_n}^{(k)} = \mathrm{E}(R_n^k),\ k \in \Re^+$. Then,

$$\mu_{R_n}^{(k)} = \mu_k(\theta_2, \alpha_2) + \lambda\left[\mathcal{S}^\star(p_1, q_1) - (1+p_1q_1)\mathcal{S}^{\star\star}(p_1, q_1)\right]\mathcal{D}(k; p_2, q_2). \tag{4.2}$$

Proof. Clearly, (4.2) is a simple consequence of (4.1). Therefore, we have only to prove the relation (4.1). Now, we have

$$\mathcal{R}_{[n]}(y) = f_Y(y) + \lambda(1-F_Y^{p_2}(y))^{q_2-1}[1-(1+p_2q_2)F^{p_2}(y)]$$

$$\times \int_0^\infty (1-F_X^{p_1}(x))^{q_1-1}[1-(1+p_1q_1)F_X^{p_1}(x)]\frac{(-\log(1-F_X(x)))^{n-1}}{\Gamma(n)}f_X(x)dx$$

$$= f_Y(y) + \sum_{i=0}^{\aleph(q_2-1)} \binom{q_2-1}{i} (-1)^i \left\{ \frac{1}{ip_2+1} f_{U_i}(y) - \frac{1+p_2q_2}{(i+1)p_2+1} f_{V_i}(y) \right\}$$

$$\sum_{j=0}^{\aleph(q_1-1)} \binom{q_1-1}{j} (-1)^j \frac{1}{\Gamma(n)} \int_0^\infty F_X^{jp_1}(x)(-\log(1-F_X(x)))^{n-1} f_X(x)dx$$

$$-(1+p_1q_1) \sum_{j=0}^{\aleph(q_1-1)} \binom{q_1-1}{i} (-1)^j \int_0^\infty F_X^{(j+1)p_1}(x)(-\log(1-F_X(x)))^{n-1} f_X(x)dx.$$

Upon using the transformation $-\log(1-F_X(x)) = t$ in the above two integrations and applying the binomial theorem on the terms $(1-e^{-t})^{jp_1}$ and $(1-e^{-t})^{(j+1)p_1}$, in the first and second integrations, respectively, we get the representation (4.1). □

Corollary 4.1. For $q_1 = q_2 = 1$ and $p_1 = p_2 = p$, i.e., for HK-FGM-GE family, we get

$$\mu_{R_n}^{(k)} = \mu_k(\theta_2, \alpha_2) + \lambda \left[\mathcal{S}^\star(p,1) - (1+p)\mathcal{S}^{\star\star}(p,1)\right] \mathcal{D}(k; p, q)$$

$$= \mu_k(\theta_2,\alpha_2) + \lambda[\mu_k(\theta_2,\alpha_2) - (1+p)\mu_k(\theta_2,\alpha_2 p)][1-(1+p)\sum_{\ell=0}^{\aleph(p_1)} (-1)^\ell \frac{\binom{p_1}{\ell}}{(\ell+1)^n}].$$

Moreover, For $q_1 = q_2 = 1$ and $p_1 = p_2 = 1$, i.e., for FGM-GE family, we get

$$\mu_{R_n}^{(k)} = \mu_k(\theta_2, \alpha_2) + \lambda \left[\mathcal{S}^\star(1,1) - (1+p)\mathcal{S}^{\star\star}(1,1)\right] \mathcal{D}(k; 1, 1)$$

$$= \mu_k(\theta_2, \alpha_2) \left[1 - \lambda(2^{-(n-1)} - 1)\right].$$

Proof. The proof is obvious, since it follows after simple algebra. □

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