Concomitants of Order statistics and record values from FGM Type Bivariate-Generalized Exponential Distribution

H. M. Barakat*, E. M. Nigm, M.A. Alawady and I. A. Husseiny
Department of Mathematics, Faculty of Science, Zagazig University,
Zagazig, Egypt

Abstract

We introduce the successive iterations in the original FGM type bivariate-generalized exponential distribution. Some distributional properties of concomitants of order statistics as well as record values for this family are studied. Recurrence relations between single as well as product moments of concomitants are obtained.

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Key Words: Concomitants; Order statistics; Record values; Generalized exponential distribution; Huang-Kotz FGM family

1 Introduction

Ordered random variables(rv's) have attracted many researshers due to their applicability in many practical areas, like order statistics. These variables occur as a natural choice when dealing with floods, drought, earthquakes, etc. The use of order statistics also appears when dealing with records. Both of order statistics and record values are used extensively in statistical models and inference, where they describe random variables arranged in order magnitude. Recored values arise naturally in many real life applications involving data related to sport, weather, and life testing studies. The record statistics provide the information about the maximum(minimum) value among all previously recorded observation, for more detail see Arnold et al. (1998). The conept of concomitant, also, called induced order statistics, arise when one sorts the members of a random sample according to corresponding values of an other random sample. The term concomitant of order statistics was first induced and applied extensively by David (1973). According to Hanif (2007)

[0]E-mail*:hbarakat2@hotmail.com

in collecting any data for an observation, several characteristics are often recorded, some of them are considered as primary and others can be observed from the primary data automatically. The later one is called concomitant or going along with variables, explanatory variables or covariables. For more detail see David and Nagaraja (1998), (2003). The most important use of concomitants of record values arises in experiments in which a specified characteristic's measurements of an individual are made sequentially, and only values that exceed or fall below the current extreme value are recorded, so that only obsevations are bivariate record values, i.e., records and their concomitants. Some properties from concomitants of record values were discussed in Ahsanullah (2009) and Ahsanullah and shakil (2013). The so-called (bivariate) Farlie-Gumbel-Morgenstern(FGM) was orignally introduced by Morgenstern (1956) for cauchy marginals. In 1960 Gumbel investigated the same structure for exponential marginals. Also, in 1960, Farlie, in connection with his investigations of the correlation coefficient, suggested a generalization of the bivariate form studied by Morgenstern and Gumbel. Huang and Kotze (1984) introduced the distribution function(df) of the classical FGM'as:

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)\left[1+\lambda\bar{F}_X(x)\bar{F}_Y(y)+\gamma F_X(x)F_Y(y)\bar{F}_X(x)\bar{F}_Y(y)\right], \qquad (1.1)$$

denoted by FGM(λ,γ).

The corresponding probability density functions (pdf) is given by :

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)\left[1+\lambda(1-2F_X(x))(1-2F_Y(y))+\gamma F_X(x)F_Y(y)(2-3F_X(x))(2-3F_Y(y))\right], \qquad (1.2)$$

where $F_X(x)$ and $F_Y(y)$ are df's, while $f_X(x)$ and $f_Y(y)$ are pdf's of the random variables (rv's) X and Y, respectively. The admissible range of the associated parameters λ and γ are $-1 \leq \lambda \leq 1$, $\lambda+\gamma \geq -1$and $\gamma \leq \frac{3-\lambda+\sqrt{9-6\lambda-3\lambda^2}}{2}$. When the marginals are uniform then, while for the classical FGM the correlation between components does not exceed $\frac{1}{3}$, the modified version HK-FGM allows correlation up to 0.39. The classical FGM distribution is a flexible family useful in applications provided that the correlation between the variables is not too large. It can be utilized for arbitrary continious marginals.

Huang and Kotz (1999) introduced a new modification of classical FGM distribution introducing additional parameters. The new Huang-Kotz FGM (HK-FGM) distributions allow correlation higher than the classical FGM and because of the simple analytical form aroused interest of many researchers. Bairamove et al. (2001) cosidered a most general form of FGM model by presenting a four-parameter extention of the classical FGM family of distributions which allow to increase the dependence between the variables.

The generalized exponential distribution (GE), a most attractive generalization of the exponential distribution, introduced by Gupta and Kundu (1999), has widespread interest and applications. The GE is an important special case the so-called class of beta generated distributions.

A continious random variable is said to be has the GE with scale parameter $\theta > 0$ and shape parameter $\alpha > 0$ (denoted by GE($\theta;\alpha$)), if the pdf and the corresponding df are given, for $x > 0$, respectively, by

$$f_X(x) = \alpha\theta(1 - \exp(-\theta x))^{\alpha-1}\exp(-\theta x), \tag{1.3}$$

and

$$F_X(x) = (1 - \exp(-\theta x))^{\alpha}, \tag{1.4}$$

Gupta and kundu (1999) showed that the kth moment of $GE(\theta;\alpha)$ is

$$\mu_k = \frac{\alpha k!}{\theta^k}\sum_{i=0}^{\aleph(\alpha-1)}\frac{(-1)^i}{(i+1)^{k+1}}\binom{\alpha-1}{i}, \tag{1.5}$$

where $\aleph(x) = \infty$, if x is non-integer and $\aleph(x) = x$, if x is integer. Moreover, the mean, variance and moment generating function of $GE(\theta;\alpha)$ are given, respectively, by

$$\mu_1 = \mathrm{E}(X) = \frac{B(\alpha)}{\theta}, \quad \mathrm{Var}(X) = \frac{C(\alpha)}{\theta^2}, \quad M_X(t) = \alpha\beta(\alpha, 1 - \frac{t}{\theta}), \tag{1.6}$$

where $B(\alpha) = \Psi(\alpha+1) - \Psi(1)$, $C(\alpha) = \Psi'(1) - \Psi'(\alpha+1)$, $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $\Psi(.)$ is the digamma function, while $\Psi'(.)$ is its derivation (the trigamma function). Recently, Tahmasebi and Jafari (2015) studied some properties of the Morgenstern type bivariate-generalized exponential distribution (denoted by MTBGED). Also, they studied some distributional properties of concomitants of order statistics as well as record values of this df. Moreover, they obtained some recurrence relations between moments of concomitants of order statistics.

In this paper, all the results of Tahmasebi and Jafari (2015) are extended to HK-FGM family with two marginals F_X and F_Y, where $X \sim GE(\theta_1;\alpha_1)$ and $Y \sim GE(\theta_2;\alpha_2)$ (denoted by HK-FGM-GE($\theta_1,\alpha_1;\theta_2,\alpha_2$)). Moreover, some new results, which were not obtained by Tahmasebi and Jafari (2015) for FGM family, are given such as recurrence relations for the single, as well as the product, moments of bivariate concomitants of order statistics, the concomitant rank-order statistics, and the asymptotic behavior of the concomitants of order statistics.

It is worth mentioning that, some of the results presented in this paper are related to paper of Beg and Ahsanullah (2008). Namely, Beg and Ahsanullah (2008) considered concomitants of generalized order statistics for the FGM family and derived the joint distribution of concomitants of two generalized order statistics and obtain their product moments. Tahmasebi and Behboodian (2012), Tahmasebi et al. (2015), Tahmasebi and Jafari (2014) and Tahmasebi et al. (2016) are further recent relevant works on this subject. Many authors studied various properties of the GE. see for example, AL-Hussaini and Ahsanullah (2015), Ahsanullah et al. (2013).

Barakat et al. (2017a) had studied the distributional properties of concomitants of order statistics as well as record values, which are belonging to the Huang-kotz Morgenstern type bivariate-generalized exponential distribution. They also, derived some general recurrence relations between single as well as product moments of concomitants. Barakat et al. (2017b) extended this study to a more general fram work by studying the distributional properties of concomitants of order statistics as well as record values from the Bairamov-Kotz-Becki-FGM type bivariate-generalized exponential distribution.

2 The Classical FGM-GE and Some of its Properties

The joint df and pdf of (X, Y) are defined by (1.1) and (1.2), respectively, where $X \sim GE(\theta_1; \alpha_1)$ and $Y \sim GE(\theta_2; \alpha_2)$. Therefore, it is easy to show that the (n, m)th joint moments of classical-FGM-GE$(\theta_1, \alpha_1; \theta_2, \alpha_2)$ is given by

$$\mathrm{E}(X^nY^m) = \mathrm{E}(X^n)\mathrm{E}(Y^m)+\lambda(\mathrm{E}(X^n)-\mathrm{E}(U_1^n))(\mathrm{E}(Y^m)-\mathrm{E}(V_1^m))+\gamma(\mathrm{E}(U_1^n)-\mathrm{E}(U_2^n))(\mathrm{E}(V_1^m)-\mathrm{E}(V_2^m)),$$

$$n, m = 1, 2, ... \qquad (2.1)$$

where $U_1 \sim GE(\theta_1; 2\alpha_1)$, $U_2 \sim GE(\theta_1; 3\alpha_1)$, $V_1 \sim GE(\theta_2; 2\alpha_2) and\ V_2 \sim GE(\theta_2; 3\alpha_2)$

Thus, by combining (2.1) and (1.3), we get

$$E(XY) = \frac{B(\alpha_1)B(\alpha_2) + \lambda D(2\alpha_1)D(2\alpha_2) + \gamma D(3\alpha_1)D(3\alpha_2)}{\theta_1\theta_2},$$

where $D((k+1)\alpha) = B(((k+1)\alpha))-B(k\alpha)$, $k = 1, 2$ Therefore, the coefficient of correlation between X and Y is

$$\rho_{X,Y} = \frac{\lambda D(2\alpha_1)D(2\alpha_2) + \gamma D(3\alpha_1)D(3\alpha_2)}{\sqrt{C(\alpha_1)C(\alpha_2)}} = \lambda g_1(\alpha_1, \alpha_2) + \gamma g_2(\alpha_1, \alpha_2).$$

Clearly, the function $g_1(\alpha_1, \alpha_2) and\ g_2(\alpha_1, \alpha_2,)$ is increasing and positive function with respect to each of $\alpha_i, i = 1, 2$.

Therefore, if λ and $\gamma > 0$, then $\rho_{X,Y}$ is increasing and positive function and if $\lambda < 0$, then $\rho_{X,Y}$ is decreasing and negative function with respect to each of α_1 and α_2. Moreover, we can show that

$\lim_{\substack{\alpha_1\to\infty\\ \alpha_2\to\infty}} g_1(\alpha_1,\alpha_2)=\frac{6(\log(2))^2}{\pi^2}$, $\quad \lim_{\substack{\alpha_1\to\infty\\ \alpha_2\to\infty}} g_2(\alpha_1,\alpha_2,)=\frac{6(\log(\frac{3}{2}))^2}{\pi^2}$,

$\lim_{\substack{\alpha_1\to 0^+\\ \alpha_2\to 0^+}} g_1(\alpha_1,\alpha_2)=0$ $\quad$ and $\lim_{\substack{\alpha_1\to 0^+\\ \alpha_2\to 0^+}} g_1(\alpha_1,\alpha_2)=0$

Therefor;

$\max\rho_{X,Y}=0.392$ at corner point $(\lambda,\gamma)=(1,1)$ and

$\min\rho_{X,Y}=-0.292$ at corner point $(\lambda,\gamma)=(-1,0)$.

The conditional df of Y given $X=x$ is given by

$$F_{Y|X}(y|x)=F_Y(y)\left[1+\lambda(1-F_Y(y))(1-2F_X(x))-\gamma F_X(x)F_Y(y)(1-F_Y(y))(2-3F_X(x))\right]. \tag{2.2}$$

Therefore, the regression curve of Y given $X=x$ for HK-FGM-GE is

$$\begin{aligned}\mathrm{E}(Y|X=x)&=\mathrm{E}(Y)+\lambda(1-2F_X(x))(\mathrm{E}(Y)-\mathrm{E}(V_1))+\gamma F_X(x)(2-3F_X(x))(\mathrm{E}(V_1)-\mathrm{E}(V_2))\\&=\frac{1}{\theta_2}[B(\alpha_2)+\lambda D(2\alpha_2)(2F_X(x)-1)+\gamma F_X(x)D(3\alpha_2)(2F_X(x)-2)],\end{aligned}$$

where $V_1\sim GE(\theta_2;2\alpha_2)$ and $V_2\sim GE(\theta_2;3\alpha_2)$ and the conditional expectation is non-linear with respect to x.

3 Concomitants of Order Statistics Based on HK-FGM-GE

Suppose $(X_i,Y_i), i=1,2,...,n$ is a random sample from a bivariate df $F_{X,Y}(x,y)$. If we order the sample by the $X-$variate, and obtain the order statistics, $X_{1:n}\le X_{1:n}\le ....\le X_{n:n}$, for the X sample, then the $Y-$variate associated with the rth order statistic $X_{r:n}$ is called the concomitant of the rth order statistic, and is denoted by $Y_{[r:n]}$.

Let $X\sim GE(\theta_1;\alpha_1)$ and $Y\sim GE(\theta_2;\alpha_2)$. Since the conditional pdf of $Y_{[r:n]}$ given $X_{[r:n]}=x$ is $f_{Y_{[r:n]}|X_{r:n}}(y|x)=f_{Y|X}(y|x)$ (cf. Galambos, 1987, see also Tahmasebi and Jafari, 2015), then the pdf of $Y_{[r:n]}$ is given by

$$f_{[r:n]}(y)=f_Y(y)+[\lambda(f_Y(y)-f_{V_1}(y))+\gamma(f_{V_2}(y)-f_{V_1}(y))]\,\Delta^{(1)}_{r,n}+[\gamma(f_{V_1}(y)-f_{V_2}(y))]\,\Delta^{(2)}_{r,n},\ y>0, \tag{3.1}$$

where

$$\Delta^{(i)}_{r,n}=\frac{\beta(r,n-r+1)-(i+1)\beta(r+i,n-r+1)}{\beta(r,n-r+1)}, i=1,2$$

Therefore, the moment generating function of $Y_{[r:n]}$ is given by

$$M_{[r:n]}(t)=\alpha_2[\beta\left(\alpha_2,1-\frac{t}{\theta_2}\right)+\lambda\Delta^{(1)}_{r,n}\left(\beta\left(\alpha_2,1-\frac{t}{\theta_2}\right)-\beta\left(2\alpha_2,1-\frac{t}{\theta_2}\right)\right)$$

$$+\gamma\Delta_{r,n}^{(2)}\left(\beta\left(2\alpha_2,1-\frac{t}{\theta_2}\right)-\beta\left(3\alpha_2,1-\frac{t}{\theta_2}\right)\right)]$$

kth moment of $Y_{[r:n]}$ is given by

$$\mu_{[r:n]}^{(k)}=\mathrm{E}[Y_{[r:n]}^k]=\mathrm{E}[Y^k]+\Delta_{r,n}^{(1)}(\gamma((\mathrm{E}[V_2^k]-\mathrm{E}[V_1^k])-\lambda(\mathrm{E}[V_1^k]-\mathrm{E}[Y^k]))-\gamma\Delta_{r,n}^{(2)}(\mathrm{E}[V_2^k]-\mathrm{E}[V_1^k])$$

putting $k=1$ we get the mean of $Y_{[r:n]}$

$$\mu_{[r:n]}=\frac{1}{\theta_2}\left[B(\alpha_2)+\Delta_{r,n}^{(1)}(\gamma D(3\alpha_2)-\lambda D(2\alpha_2))-\gamma\Delta_{r,n}^{(2)}D(3\alpha_2)\right]. \tag{3.2}$$

Theorem 3.1. for any $1\le r\le n-3$, we get

$$(B-3(r+1))\mu_{[r+2:n]}=(2B-3(2r+3))\mu_{[r+1:n]}-(B-3(r-2))\mu_{[r:n]}$$

Moreover , for all $n>2$,we get

$$(A^*(2-n(n+1))+3(r+1)(1-n))\mu_{[r:n]}=(n+2)(A^*(n+1)+3(r+1))+3(r+1))\mu_{[r:n-2]}$$
$$-(2A^*(n+2)+3(r+1)(2n+1))\mu_{[r:n-1]}$$

where$B=(n+2)A^*$and$A^*=1-\frac{\lambda D(2\alpha_2)}{\gamma D(3\alpha_2)}$

Proof of Theorem 3.1.

$$\frac{\mu_{[r+2:n]}-\mu_{[r:n]}}{\mu_{[r+1:n]}-\mu_{[r:n]}}=\frac{A^*(\Delta_{r+2,n}^{(1)}-\Delta_{r,n}^{(1)})+\Delta_{r+2,n}^{(2)}-\Delta_{r,n}^{(2)}}{A^*(\Delta_{r+1,n}^{(1)}-\Delta_{r,n}^{(1)})+\Delta_{r+1,n}^{(2)}-\Delta_{r,n}^{(2)}}$$

We can check that

$$\Delta_{r+2,n}^{(1)}-\Delta_{r,n}^{(1)}=\frac{-4}{n+1},\quad \Delta_{r+1,n}^{(1)}-\Delta_{r,n}^{(1)}=\frac{-2}{n+1}$$

and

$$\Delta_{r+2,n}^{(2)}-\Delta_{r,n}^{(2)}=\frac{-12r-18}{(n+1)(n+2)},\quad \Delta_{r+1,n}^{(2)}-\Delta_{r,n}^{(2)}=\frac{-6r-6}{(n+1)(n+2)}$$

Also, We can check that

$$\Delta_{r,n}^{(1)}-\Delta_{r,n-2}^{(1)}=\frac{4r}{(n-1)(n+1)},\quad \Delta_{r,n-1}^{(1)}-\Delta_{r,n-2}^{(1)}=\frac{2r}{n(n-1)}$$

and

$$\Delta^{(2)}_{r,n} - \Delta^{(2)}_{r,n-2} = \frac{6r(r+1)(2n+1)}{n(n-1)(n+1)(n+2)}, \quad \Delta^{(2)}_{r,n-1} - \Delta^{(1)}_{r,n-2} = \frac{6r(r+1)}{n(n-1)(n+1)}$$

□

By multiplying the both sides of (3.1) by $(y-\mu_{[r:n]})^2$ and integrating, we obtain the variance of $Y_{[r:n]}$ as

$$\sigma^2_{[r:n]} = \frac{1}{\theta_2^2}[(1+\delta r_1)(C(\alpha_2)-\delta r_1 B^2(2\alpha_2)) + (\delta r_2 - \delta r_1)(C(2\alpha_2)+B^2(2\alpha_2))$$
$$-B^2(2\alpha_2)(\delta r_1+\delta r_2)^2 + \delta r_2(C(3\alpha_2) - B^2(3\alpha_2)(1+\delta r_2)) - 2B(\alpha_2)B(2\alpha_2)\delta r_3$$
$$-2B(\alpha_2)B(3\alpha_2)\delta r_2(1+\delta r_1) - 2B(2\alpha_2)B(3\alpha_2)\delta r_2(\delta r_1+\delta r_2)]. \quad (3.3)$$

where $\delta r_1 = \lambda\Delta^{(1)}_{r,n}$, $\delta r_2 = \gamma\Delta^{(2)}_{r,n}$ and $\delta r_3 = \delta r_1(1+\delta r_1) + \delta r_2(1+\delta r_1)$

It is well-known that in many cases, the concomitants of the extremes among the X's are not extremes among the Y's (with high probability) (cf. Galambos, 1987). This fact aroused interest of some researchers to investigate the rank (of $Y_{[r:n]}$) $\mathcal{R}_{[r:n]} = \sum\limits_{j=1}^{n} \mathrm{I}(Y_{[r:n]} - Y_j)$, where $\mathrm{I}(x)=1$, if $x \geq 0$, $\mathrm{I}(x)=0$, if $x<0$. The distribution of $R_{r:n}$ is obtained by David et al. (1977). Barakat and El-Shandidy (2004) gave a new representation of the df and the expected value of $\mathcal{R}_{[r:n]}$. Namely, for all $r,s=2,3,...,n-1$, we have

$$A_{r:n}(s) = P(\mathcal{R}_{[r:n]}=s) = n[\mathrm{E}(\mathcal{C}(W_{r:n-1},Z_{s:n-1})) - \mathrm{E}(\mathcal{C}(W_{r-1:n-1},Z_{s:n-1}))$$
$$- \mathrm{E}(\mathcal{C}(W_{r:n-1},Z_{s-1:n-1})) + \mathrm{E}(\mathcal{C}(W_{r-1:n-1},Z_{s-1:n-1}))], \quad (3.4)$$

where $\mathcal{C}(.,.)$ is the copula of the bivariate df $F_{X,Y}(x,y)$, i.e., $\mathcal{C}(w,z) = wz(1+\lambda(1-w^p)(1-z^p))$. Moreover, $W_{j:n}=F_X(X_{j:n})$ and $Z_{j:n}=F_Y(Y_{j:n})$ are the jth uniform order statistics with expectation $\mathrm{E}(W_{j:n}) = \mathrm{E}(Z_{j:n}) = \frac{j}{n+1}$. The representation (3.4) enables us to use the $\delta-$method (with one-step Taylor approximation) to compute an approximate formula for the df $A_{r:n}(s)$, by

$$A_{r:n}(s) \sim n\left[\mathcal{C}(\frac{r}{n},\frac{s}{n}) - \mathcal{C}(\frac{r-1}{n},\frac{s}{n}) - \mathcal{C}(\frac{r}{n},\frac{s-1}{n}) + \mathcal{C}(\frac{r-1}{n},\frac{s-1}{n})\right] = \frac{1+\lambda}{n}$$
$$-\frac{\lambda}{n^{p+1}}[rs(r^p+s^p)-(r-1)s((r-1)^p+s^p)-r(s-1)(r^p+(s-1)^p)+(r-1)(s-1)((r-1)^p+(s-1)^p)]$$
$$+\frac{\lambda}{n^{2p+1}}[r^{p+1}s^{p+1} - (r-1)^{p+1}s^{p+1} - r^{p+1}(s-1)^{p+1} + (r-1)^{p+1}(s-1)^{p+1}].$$

The limiting distribution of $Y_{[n:n]}$, as $n\to\infty$, depends on the conditional distribution of Y given X and the marginal distribution of X, and it is given by the following theorem.

3.1 Joint Distribution of Concomitants of Order Statistics Based on HK-FGM-GE

The joint pdf of concomitants $Y_{[r:n]}$ and $Y_{[s:n]}, r < s$, is (cf. Tahmasebi and Jafari, 2015)

$$f_{[r,s:n]}(y_1, y_2) = \int_0^\infty \int_0^{x_2} f_{Y|X}(y_1|x_1) f_{Y|X}(y_2|x_2) f_{r,s:n}(x_1, x_2) dx_1 dx_2,$$

where $\beta(a, b, c) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$ and

$$f_{r,s:n}(x_1, x_2) = \frac{1}{\beta(r, s-r, n-s+1)} F_X^{r-1}(x_1)$$
$$\times (F_X(x_2) - F_X(x_1))^{s-r-1} (1 - F_X(x_2))^{n-s} f_X(x_1) f_X(x_2), x_1 < x_2.$$

Therefore,

$$f_{[r,s:n]}(y_1, y_2) = \int_0^\infty \int_0^{x_2} = f_Y(y_1) \left[1 + \lambda(1 - 2F_X(x_1))(1 - 2F_Y(y_1)) + \gamma F_X(x_1) F_Y(y_1)(2 - 3F_X(x_1))(2 - 3\right.$$
$$\times \left[f_Y(y_2) \left[1 + \lambda(1 - 2F_X(x_2))(1 - 2F_Y(y_2)) + \gamma F_X(x_2) F_Y(y_2)(2 - 3F_X(x_2))(2 - 3F_Y(y_2))\right]\right]$$
$$\times \left[\frac{F_X^{r-1}(x_1)(F_X(x_2) - F_X(x_1))^{s-r-1}(1 - F_X(x_2))^{n-s}}{\beta(r, s-r, n-s+1)} f_X(x_1) f_X(x_2)\right] dx_1 dx_2. \quad (3.5)$$

On the other hand, we have

$$f_{[r,s:n]}(y_1, y_2) = f_Y(y_1) f_Y(y_2)[1 + \lambda(1 - 2F_Y(y_1))I_1 + \lambda(1 - 2F_Y(y_2))I_2$$
$$+\lambda^2(1 - 2F_Y(y_1))(1 - 2F_Y(y_2))I_3 + \gamma F_Y(y_1)(2 - 3F_Y(y_1))I_4 + \gamma F_Y(y_2)(2 - 3F_Y(y_2))I_5$$
$$+\gamma^2 F_Y(y_1) F_Y(y_2)(2 - 3F_Y(y_1))(2 - 3F_Y(y_2))I_6 + \lambda\gamma F_Y(y_2)(1 - 2F_Y(y_1))(2 - 3F_Y(y_2))I_7$$
$$+\lambda\gamma F_Y(y_1))(1 - 2F_Y(y_2))(2 - 3F_Y(y_1))I_8]$$

where
$I_1 = \Delta^{(1)}_{r,s,n}$, $I_2 = \Delta^{(2)}_{r,s,n}$, $I_3 = \Delta^{(1)}_{r,s,n} + \Delta^{(2)}_{r,s,n} - \Delta^{(3)}_{r,s,n}$
,$I_4 = \Delta^{(4)}_{r,s,n} - \Delta^{(1)}_{r,s,n}$ $I_5 = \Delta^{(5)}_{r,s,n} - \Delta^{(2)}_{r,s,n}$
$I_6 = (\Delta^{(6)}_{r,s,n} + \Delta^{(7)}_{r,s,n}) - (\Delta^{(3)}_{r,s,n} - \Delta^{(8)}_{r,s,n})$
$I_7 = \Delta^{(5)}_{r,s,n} - \Delta^{(2)}_{r,s,n} + \Delta^{(3)}_{r,s,n} - \Delta^{(7)}_{r,s,n}$
$I_8 = \Delta^{(4)}_{r,s,n} - \Delta^{(1)}_{r,s,n} + \Delta^{(3)}_{r,s,n} - \Delta^{(6)}_{r,s,n}$
by using some algebra we can check that :

$$\Delta^{(i)}_{r,s,n} = \frac{\beta(r, s-r, n-s+1) - (p+1)\beta(r+p, s-r, n-s+1)}{\beta(r, s-r, n-s+1)}, (i = 1 : p = 1), (i = 4 : p = 2)$$
$$\Delta^{(i)}_{r,s,n} = \frac{\beta(r, s-r, n-s+1) - (p+1)\beta(s+p, n-s+1)\beta(r, s-r)}{\beta(r, s-r, n-s+1)}, (i = 2 : p = 1), (i = 5 : p = 2)$$

$$\Delta^{(i)}_{r,s,n} = \frac{\beta(r, s-r, n-s+1) - (p+1)^2\beta(s+2p, n-s+1)\beta(r+p, s-r)}{\beta(r, s-r, n-s+1)}, (i = 3 : p = 1), (i = 8 : p = 2)$$

$$\Delta^{(i)}_{r,s,n} = \frac{\beta(r, s-r, n-s+1) - 6\beta(s+3, n-s+1)\beta(r+p, s-r)}{\beta(r, s-r, n-s+1)}, (i = 7 : p = 1), (i = 6 : p = 2)$$

The product moment $\mathrm{E}[Y_{[r:n]}Y_{[s:n]}]$ is obtained directly as

$$\begin{aligned}
\mu_{[r,s:n]} &= \frac{1}{\theta_2^2}[B^2(\alpha_2)\xi_1(\lambda, r, s, n) - B(\alpha_2)B(2\alpha_2)\xi_2(\gamma, \lambda, r, s, n) \\
&+ B^2(2\alpha_2)\xi_3(\gamma, \lambda, r, s, n) - B(\alpha_2)B(2\alpha_2)\xi_4(\gamma, \lambda, r, s, n) \\
&+ B^2(3\alpha_2)\gamma^2 I_6]. \qquad (3.6)
\end{aligned}$$

where

$$\begin{aligned}
\xi_1(\lambda, r, s, n) &= 1 + \lambda(I_1 + I_2 + I_3), \\
\xi_2(\gamma, \lambda, r, s, n) &= \lambda(I_1 + I_2 + 2\lambda I_3) - \gamma(I_4 + I_5) - \lambda\gamma(I_7 + I_8), \\
\xi_3(\gamma, \lambda, r, s, n) &= \lambda^2 I_3 + \gamma^2 I_6 - \lambda\gamma(I_7 + I_8), \\
\xi_4(\gamma, \lambda, r, s, n) &= \gamma(I_4 + I_5) + \lambda\gamma(I_7 + I_8).
\end{aligned}$$

Therefore, by using (3.2) and (3.6) we can after some algebra calculate the covariance between $Y_{[r:n]}$ and $Y_{[s:n]}$ as

$$\begin{aligned}
\sigma_{[r,s:n]} &= \frac{1}{\theta_2^2}[B^2(\alpha_2)\delta^{(1)}_{r,s,n} - B(\alpha_2)B(2\alpha_2)\delta^{(2)}_{r,s,n} + B^2(2\alpha_2)\delta^{(3)}_{r,s,n} \\
&- B(\alpha_2)B(3\alpha_2)\delta^{(4)}_{r,s,n} + B^2(3\alpha_2)\delta^{(5)}_{r,s,n}]. \qquad (3.7)
\end{aligned}$$

where

$$\begin{aligned}
\delta^{(1)}_{r,s,n} &= 1 + \lambda(I_1 + I_2 + \lambda I_3 - \Delta^{(1)}_{r,n} - \Delta^{(1)}_{s,n}), \\
\delta^{(2)}_{r,s,n} &= \lambda(I_1 + I_2 + 2\lambda I_3 - \Delta^{(1)}_{r,n} - \Delta^{(1)}_{s,n}) - \gamma(I_4 + I_5 - \Delta^{(2)}_{r,n} - \Delta^{(2)}_{s,n}) - \lambda\gamma(I_7 + I_8), \\
\delta^{(3)}_{r,s,n} &= \lambda^2(I_3 + \Delta^{(1)}_{r,n}\Delta^{(1)}_{s,n}) + \gamma^2(I_6 + \Delta^{(2)}_{r,n}\Delta^{(2)}_{s,n}) - \lambda\gamma(I_7 + I_8), \\
\delta^{(4)}_{r,s,n} &= \gamma(I_4 + I_5 - \Delta^{(2)}_{r,n}\Delta^{(2)}_{s,n}) + \lambda\gamma(I_7 + I_8), \\
\delta^{(5)}_{r,s,n} &= \gamma^2(I_6 + \Delta^{(2)}_{r,n}\Delta^{(2)}_{s,n}).
\end{aligned}$$

We can now use (3.7) and (3.3) to obtain the coefficient of correlation between $Y_{[r:n]}$ and $Y_{[s:n]}$ as

$$\rho_{[r,s:n]} = \frac{\sigma_{[r,s:n]}}{\sqrt{\sigma^2_{[r:n]}\sigma^2_{[s:n]}}}, \qquad (3.8)$$

$\mu_{[r,s:n]}$.

Theorem 3.3. For any $1 \leq r \leq n-3$, we get

$$\mu_{[r+2,s:n]} = 2\mu_{[r+1,s:n]} - \mu_{[r,s:n]} - \tau_n(s, \alpha_2, \lambda, \gamma). \tag{3.9}$$

where$\tau_n(s, \alpha_2, \lambda, \gamma) = \frac{6A_1(n+3)(n+4)+12A_2(s+2)(n+4)+18A_3(s+2)(s+3)}{(n+1)(n+2)(n+3)(n+4)}$,

Moreover, $1 \leq s \leq n-3$, we get

$$\mu_{[r,s+2:n]} = 2\mu_{[r,s+1:n]} - \mu_{[r,s:n]} - \omega_n(r, \alpha_2, \lambda, \gamma). \tag{3.10}$$

where$\omega_n(r, \alpha_2, \lambda, \gamma) = \frac{6A_4(n+3)(n+4)+12rA_5(n+4)+18A_3r(r+1)}{(n+1)(n+2)(n+3)(n+4)}$,

Finally, for all $n > 2$, we get

$$(n+1)\mu_{[r,s:n]} = 2n\mu_{[r,s:n-1]} - (n-1)\mu_{[r,s:n-2]} + \zeta_n(r, n, s\alpha_2, \lambda, \gamma), \tag{3.11}$$

where $\zeta_n(r, n, s\alpha_2, \lambda, \gamma) = \frac{6A_4s(s+1)(n+3)(n+4)+18rA_2r(r+1)(s+2)(n+4)+18A_5(s+1)(s+2)(n+4)+36A_3(s+2)(s+3)r(r+1)}{n(n+1)(n+2)(n+3)(n+4)}$,
$A_1 = \frac{1}{\theta_2^2}[\lambda\gamma(B^2(2\alpha_2) + B(\alpha_2)B(2\alpha_2) - B(\alpha_2)B(3\alpha_2)) - \gamma B(\alpha_2)B(2\alpha_2)]$,
$A_2 = \frac{1}{\theta_2^2}[\gamma^2(B^2(2\alpha_2) + B^2(3\alpha_2)) + \lambda\gamma(B(\alpha_2)B(3\alpha_2) - B(\alpha_2)B(2\alpha_2))]$,
$A_3 = \frac{-\gamma^2}{\theta_2^2}(B^2(2\alpha_2) + B^2(3\alpha_2))$.
$A_4 = \frac{1}{\theta_2^2}[\gamma(B(\alpha_2)B(2\alpha_2) - B(\alpha_2)B(3\alpha_2)) + \lambda\gamma(B(\alpha_2)B(2\alpha_2) + B(\alpha_2)B(3\alpha_2) - B^2(\alpha_2))]$,
$A_5 = \frac{1}{\theta_2^2}[\gamma^2(B^2(2\alpha_2) + B^2(3\alpha_2)) + \lambda\gamma(B(\alpha_2)B(3\alpha_2) - B(\alpha_2)B(2\alpha_2) + B^2(\alpha_2))]$,

Proof. It is easy to check that

$$\Delta^{(i)}_{r+2,s,n} - \Delta^{(i)}_{r,s,n} = 2(\Delta^{(i)}_{r+1,s,n} - \Delta^{(i)}_{r,s,n}), i = 1, 3, 6 \tag{3.12}$$

$$\Delta^{(i)}_{r+2,s,n} - \Delta^{(i)}_{r,s,n} = (\Delta^{(i)}_{r+1,s,n} - \Delta^{(i)}_{r,s,n})\frac{2r+3}{r+1}, i = 4, 7, 8 \tag{3.13}$$

and

$$\Delta^{(2)}_{r,s,n} = \Delta^{(2)}_{r+1,s,n} = \Delta^{(2)}_{r+2,s,n}. \tag{3.14}$$

$$\Delta^{(5)}_{r,s,n} = \Delta^{(5)}_{r+1,s,n} = \Delta^{(5)}_{r+2,s,n}. \tag{3.15}$$

The recurrence relation (3.9) is now followed by combining (3.12), (3.13) and (3.14) with (3.14). Now, we turn to prove (3.10). First, we notice that

$$\Delta^{(1)}_{r,s,n} = \Delta^{(1)}_{r,s+1,n} = \Delta^{(1)}_{r,s+2,n}. \tag{3.16}$$

and

$$\Delta^{(4)}_{r,s,n} = \Delta^{(4)}_{r,s+1,n} = \Delta^{(4)}_{r,s+2,n}. \tag{3.17}$$

Moreover, it is easy to check that

$$\Delta^{(i)}_{r,s+2,n} - \Delta^{(i)}_{r,s,n} = 2(\Delta^{(i)}_{r,s+1,n} - \Delta^{(i)}_{r,s,n}), i = 2, 3, 6, \tag{3.18}$$

$$\Delta^{(i)}_{r,s+2,n} - \Delta^{(i)}_{r,s,n} = (\Delta^{(i)}_{r,s+1,n} - \Delta^{(i)}_{r,s,n})\frac{2s+2p+1}{s+p}, i = 5 : p = 1, i = 7 : p = 2 \quad and \quad i = 8 : p = 3 \tag{3.19}$$

Therefore, the recurrence relation (3.10) is now followed by combining (3.16),(3.17),(3.18)with(3.19) In order to prove the recurrence relation (3.11), we first notice that

$$\Delta^{(i)}_{r,s,n-2:p} - \Delta^{(i)}_{r,s,n:p} = (\Delta^{(i)}_{r,s,n-1:p} - \Delta^{(i)}_{r,s,n:p})\frac{2n+p-1}{n-1}, i = 1, 2, : p = 1 \quad and \quad i = 4, 5 : p = 2 \tag{3.20}$$

and

$$\Delta^{(i)}_{r,s,n-2:p} - \Delta^{(i)}_{r,s,n:p} = (\Delta^{(i)}_{r,s,n-1:p} - \Delta^{(i)}_{r,s,n:p})\frac{2n+2p-1}{n-1}, i = 6, 7 : p = 1 \quad and \quad i = 8 : p = 2. \tag{3.21}$$

The recurrence relation (3.11) is now followed by combining (3.20) with (3.21). The theorem is established. □

4 Concomitants of Record Values Based on HK-FGM-GE

Let $\{(X_i, Y_i)\}, i = 1, 2, ...$ be a random sample from HK-FGM-GE$(\theta_1, \alpha_1; \theta_2, \alpha_2)$. When the experimenter interests in studying just the sequence of records of the first component X_i's, the second component associated with the record value of the first one is termed as the concomitant of that record value. The concomitants of record values arise in a wide variety of practical experiments, e.g., see Bdair and Raqab (2013) and Arnold et al. (1998). Let $\{R_n, n \geq 1\}$ be the sequence of record values in the sequence of X's, while $R_{[n]}$ be the corresponding concomitant. Houchens (1984) has obtained the pdf of concomitant of nth record value for $n \geq 1$, as $h_{[n]}(y) = \int_0^\infty f_Y(y|x)g_n(x)dx$, where $g_n(x) = \frac{1}{\Gamma(n)}(-\log(1 - F_X(x)))^{n-1}f_X(x)$ is the pdf of R_n. Therefore, after some algebra, we get

$$h_{[n]}(y) = (1 + \lambda\Upsilon_{n:1})f_Y(y) + (\gamma\Upsilon_{n:2} - \lambda\Upsilon_{n:2})f_{V_1}(y) - \gamma\Upsilon_{n:2}f_{V_2}(y), \tag{4.1}$$

where $V_1 \sim GE(\theta_2; 2\alpha_2))$,$V_2 \sim GE(\theta_2; 3\alpha_2))$and

$$\Upsilon_{n:2} = \left[1 - 3\sum_{i=0}^{\aleph(2)} \frac{(-1)^i \binom{2}{i}}{(i+1)^n}\right].$$

Clearly, $\Upsilon_{n:1} = (2^{-(n-1)} - 1)$

The representation (4.1) enables us to derive the mean and the variance of $R_{[n]}$ as

$$\mu_{[R_n]} = \frac{1}{\theta_2}\left[B(\alpha_2) - \lambda\Upsilon_{n:1}D(2\alpha_2) - \gamma\Upsilon_{n:2}D(3\alpha_2)\right]$$

and

$$\begin{aligned}\sigma^2_{[R_n]} = \frac{1}{\theta_2^2}[C(\alpha_2) - \lambda\Upsilon_{n:1} - \gamma\Upsilon_{n:2} \\ -(1+\lambda\Upsilon_{n:1})\lambda\Upsilon_{n:1}(D(2\alpha_2))^2 - (1+\gamma\Upsilon_{n:2})\gamma\Upsilon_{n:2}(D(3\alpha_2))^2 \\ +\lambda\Upsilon_{n:1}B(\alpha_2)D(2\alpha_2) + \gamma\Upsilon_{n:2}B(\alpha_2)D(3\alpha_2) - \lambda\gamma\Upsilon_{n:1}\Upsilon_{n:2}D(2\alpha_2)D(3\alpha_2)].\end{aligned} \tag{4.2}$$

which are the correction formulas of the mean and the variance, respectively, of $R_{[n]}$ given by Tahmasebi and Jafari (2015) for MTBGED.

The joint pdf of the concomitants $R_{[n]}$ and $R_{[m]}$, $n < m$, is given by

$$h_{[n,m]}(y_1, y_2) = \int_0^\infty \int_{x_1}^\infty f_{Y|X}(y_1|x_1) f_{Y|X}(y_2|x_2) g_{m,n}(x_1, x_2) dx_2 dx_1,$$

where

$$g_{m,n}(x) = \frac{1}{\Gamma(n)\Gamma(m-n)}(-\log(1-F_X(x_1)))^{n-1}\left(-\log\frac{1-F_X(x_2)}{1-F_X(x_1)}\right)^{m-n-1}\frac{f_X(x_1)f_X(x_1)}{1-F_X(x_1)}$$

is the joint pdf of R_n and R_m. Therefore, after some algebra, we get

$$\begin{aligned}h_{[n,m]}(y_1, y_2) &= f_Y(y_1)f_Y(y_2)[1 + \lambda(1-2F_Y(y_1))k_1 + \lambda(1-2F_Y(y_2))k_2 \\ &+ \lambda^2(1-2F_Y(y_1))(1-2F_Y(y_2))k_3 + \gamma F_Y(y_1)(2-3F_Y(y_1))k_4 \\ &+ \gamma F_Y(y_2)(2-3F_Y(y_2))k_5 + \gamma^2 F_Y(y_1)F_Y(y_2)(2-3F_Y(y_1))(2-3F_Y(y_2))k_6 \\ &+ \lambda\gamma F_Y(y_2)(1-2F_Y(y_1))(2-3F_Y(y_2))k_7 + \lambda\gamma F_Y(y_1))(1-2F_Y(y_2))(2-3F_Y(y_1))k_8],\end{aligned} \tag{4.3}$$

where $k_1 = \Upsilon_{(n:p=1)}$, $k_2 = \Upsilon_{(m:p=1)}$, $k_3 = \Upsilon_{(n:1)} + \Upsilon_{(m:1)} - \Upsilon_{(n,m:p=q=1)}$
$k_4 = \Upsilon_{(n:p=2)} - \Upsilon_{(n:p=1)}$, $k_5 = \Upsilon_{(m:p=2)} - \Upsilon_{(m:p=1)}$,
$k_6 = \Upsilon_{(n,m:p=2,q=1)} + \Upsilon_{(n,m:p=1,q=2)} - \Upsilon_{(n,m:p=q=1)} - \Upsilon_{(n,m:p=q=2)}$
$k_7 = \Upsilon_{(m:p=2)} + \Upsilon_{(n,m:p=q=1)} - \Upsilon_{(m:p=1)} - \Upsilon_{(n,m:p=1,q=2)}$
$k_8 = \Upsilon_{(n:p=2)} + \Upsilon_{(n,m:p=q=1)} - \Upsilon_{(n:p=1)} - \Upsilon_{(n,m:p=2,q=1)}$
by using some algebra we can check that :

$$\Upsilon_{n:p} = \left[1 - (1+p)\sum_{i=0}^{\aleph(p)} \frac{(-1)^i \binom{p}{i}}{(i+1)^n}\right],$$

$$\Upsilon_{n,m:p,q} = \left[1 - (1+p)(1+q) \sum_{i=0}^{\aleph(p)} \sum_{j=0}^{\aleph(q)} \frac{(-1)^{i+j} \binom{p}{i} \binom{q}{j}}{(i+j+1)^n (j+1)^{m-n}} \right].$$

The representation (4.3) enables us to derive the product moment and the covariance of $R_{[n]}$ and $R_{[m]}$ as

$$\begin{aligned} \mu_{[R_n,R_m]:p} &= \frac{1}{\theta_2^2}[B^2(\alpha_2)\xi_1(\lambda,n,m) - B(\alpha_2)B(2\alpha_2)\xi_2(\gamma,\lambda,n,m) \\ &+ B^2(2\alpha_2)\xi_3(\gamma,\lambda,n,m) - B(\alpha_2)B(2\alpha_2)\xi_4(\gamma,\lambda,n,m) \\ &+ B^2(3\alpha_2)\gamma^2 k_6]. \end{aligned}$$

where

$$\begin{aligned} \xi_1(\lambda,n,m) &= 1 + \lambda(k_1+k_2+k_3), \\ \xi_2(\gamma,\lambda,n,m) &= \lambda(k_1+k_2+2\lambda k_3) - \gamma(k_4+k_5) - \lambda\gamma(k_7+k_8), \\ \xi_3(\gamma,\lambda,n,m) &= \lambda^2 k_3 + \gamma^2 k_6 - \lambda\gamma(k_7+k_8), \\ \xi_4(\gamma,\lambda,n,m) &= \gamma(k_4+k_5) + \lambda\gamma(k_7+k_8). \end{aligned}$$

and

$$\begin{aligned} \sigma_{[R_n,R_m]} &= \frac{1}{\theta_2^2}[B^2(\alpha_2)\delta_{n,m}^{(1)} - B(\alpha_2)B(2\alpha_2)\delta_{n,m}^{(2)} + B^2(2\alpha_2)\delta_{n,m}^{(3)} \\ &- B(\alpha_2)B(3\alpha_2)\delta_{n,m}^{(4)} + B^2(3\alpha_2)\delta_{n,m}^{(5)}]. \qquad (4.4) \end{aligned}$$

where

$$\begin{aligned} \delta_{n,m}^{(1)} &= 1 + \lambda(k_1+k_2+\lambda k_3 - \Upsilon_{(n:p=1)} - \Upsilon_{(m:p=1)}), \\ \delta_{n,m}^{(2)} &= \lambda(k_1+k_2+2\lambda I_3 - \Upsilon_{(n:p=1)} - \Upsilon_{(m:p=1)}) - \gamma(k_4+k_5 - \Upsilon_{(n:p=2)} - \Upsilon_{(m:p=2)}) - \lambda\gamma(k_7+k_8), \\ \delta_{n,m}^{(3)} &= \lambda^2(k_3 + \Upsilon_{(n:p=1)}\Upsilon_{(m:p=1)}) + \gamma^2(k_6 + \Upsilon_{(n:p=2)}\Upsilon_{(m:p=2)}) - \lambda\gamma(I_7+I_8), \\ \delta_{n,m}^{(4)} &= \gamma(k_4+k_5 - \Upsilon_{(n:p=2)}\Upsilon_{(m:p=2)}) + \lambda\gamma(k_7+k_8), \\ \delta_{n,m}^{(5)} &= \gamma^2(k_6 + \Upsilon_{(n:p=2)}\Upsilon_{(m:p=2)}). \end{aligned}$$

Finally, combining (4.2) with (4.4), we get the correlation coefficient of the concomitants $R_{[n]}$ and $R_{[m]}$, as

$$\rho_{[R_n,R_m]} = \frac{\sigma_{[R_n,R_m]}}{\sqrt{\sigma_{[R_n]}^2 \sigma_{[R_m]}^2}},$$

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