GEHRING DYNAMIC INEQUALITIES AND HIGHER INTEGRABILITY THEOREMS OF NONINCREASING FUNCTIONS

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ABSTRACT. In this talk, I will present new dynamic inequalities based on the application of the time scale version of Hardy's type inequality on a finite interval $[a,b]_{\mathbb{T}}$ where T is a time scale. Next, we will speak about Gehring's type inequalities on time scales by employing the obtained inequality. As an application of Gehring inequalities, we will prove some interpolation. Next, we will prove a dynamic inequality of Shum's type on a time scale $\mathbb{T}$. The proof is new and different from the proof due to Shum. [Canad. Math. Bull. 14 (1971), 225-230]. Next, we prove some new integrability theorems which as a special case, when $\mathbb{T}=\mathbb{R}$, contain the results due to Muckenhoupt [Tran. Amer. Math. Soc. 165 (1972), 207-226] and the results due to Bojarski, Sbordone and Wik. [Studia Mat. VII, 10 (1992), 155-163]. By employing theorems, we will prove a higher integrability result which proves that the space $L^q_\Delta(0,T]_{\mathbb{T}}$ of nonincreasing functions will be in the space $L^p_\Delta(0,T]_{\mathbb{T}}$ for $p>q$. The results contain, as a special case, the integrability results due to Alzer [J. Math. Anal. Appl. 190 (1995), 774-779]. When $\mathbb{T}=\mathbb{N}$ our results are essentially new and can be applied on different types of time scales.

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1. INTRODUCTION

In 1972 Muckenhoupt [16] proved a sharp higher integrability result for decreasing functions from reverse mean value integral inequality as follows: Let $f(t)$ be positive and decreasing on $(0,T]\subset\mathbb{R}$ and assume that there exists $A>1$ such that

$$\frac{1}{t}\int_0^t f(s)ds \le Af(t), \quad \text{for all } t\in(0,T], \tag{1.1}$$

then, for every $p\in[1,\,A/(A-1)]$, the function f belongs to $L^p(0,T]$ and

$$\frac{1}{T}\int_0^T f^p(t)dt \le \frac{A}{A-p(A-1)}\left(\frac{1}{T}\int_0^T f(s)ds\right)^p. \tag{1.2}$$

A function verifying (1.1) is called an A_1-weight Muckenhoupt. In 1973 Gehring in [11] extended the result of Muckenhoupt and proved that if $f\in L^q(I)$, $q>1$ and satisfies the reversed Hölder's inequality

$$\left(\frac{1}{|I|}\int_I f^q dx\right)^{1/q} \le C\frac{1}{|I|}\int_I f dx, \tag{1.3}$$

for any cube I with sides parallel to the axes with measure $|I|$ and $C > 1$, then $f \in L^p(I)$, for $p > q$ and there exists $K > 1$ such that

$$\left(\frac{1}{|I|}\int_I f^p dx\right)^{1/p} \leq K\frac{1}{|I|}\int_I f dx. \tag{1.4}$$

The reverse of this inequality appears, for example, in the paper [11] by Gehring in his remarkable theorem showing higher integrability of a function verifying the reverse Hölder's inequality (1.3). Reverse integral inequalities (cf. [10, 11, 20]) and its many variants and extensions play important roles in nonlinear PDEs, in the study of weighted norm inequalities for the classical operators of harmonic analysis, as well as in functional analysis. These inequalities also appear in different fields of analysis such as quasiconformal mappings, weighted Sobolev imbedding theorem and regularity theory of variational problems (we refer the reader to the papers [14, 15]).

In 1990 Nania [17] extended the results of Muckenhoupt and Gehring and proved a higher integrability theorem for decreasing functions by using the inequality

$$\frac{1}{t}\int_0^t f^q(s)ds \leq Cf^{q-1}(t)\frac{1}{t}\int_0^t f(s)ds, \text{ for all } t \in (0, T], \tag{1.5}$$

where the constants $C > 1$ and $q > 1$. The inequality (1.5) is the converse of the inequality

$$f^{q-1}(t)\frac{1}{t}\int_0^t f(s)ds \leq \frac{1}{t}\int_0^t f^q(s)ds, \quad \text{for all } t \in (0, T], \tag{1.6}$$

which holds for all nonnegative and decreasing function $f \in L^q(0, T]$. In particular Nania proved that if (1.5) holds, then for every $p \in [q, q + \varepsilon]$ the function $f \in L^p(0, T]$ and

$$\left(\frac{1}{T}\int_0^T f^p(t)dt\right)^{1/p} \leq K\left(\frac{1}{T}\int_0^T f(s)ds\right), \tag{1.7}$$

where $\varepsilon = q/(\alpha - 1)$, $\alpha = Cq(q - 1)$,

$$K = \left[\frac{\alpha^{r+1}}{\alpha - r(\alpha - 1)}\right]^{1/p}, \text{ and } r = p/q.$$

The inequality (1.7) has been proved by employing the classical Hardy integral inequality (see [12])

$$\int_0^T \left(\frac{1}{t}\int_0^t f(s)ds\right)^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^T f^p(t)dt, \quad p > 1. \tag{1.8}$$

In 1992 Bojarski, Sbordone and Wik [8] improved the Muckenhoupt inequality by excluding the monotonicity condition on the function f with a best constant. The proof that has been given in [8] has been done by using the rearrangement of the function over the interval I. In particular, they proved that if f satisfies (1.1) with $c \geq 1$ then

$$\frac{1}{|I|}\int_I f^p(t)dt \leq \frac{c^{1-p}}{c - p(c-1)}\left(\frac{1}{|I|}\int_I f(s)ds\right)^p, \text{ for } p < c/(c-1). \tag{1.9}$$

In 1995 Alzer [5] proved a new refinement of Nania's inequality by using the following inequality

$$\int_0^T \left(\frac{1}{t}\int_0^t f(s)ds\right)^p \Delta t + \frac{p}{p-1}T^{1-p}\left(\int_0^T f(t)dt\right)^p \leq \left(\frac{p}{p-1}\right)^p \int_0^T f^p(t)dt, \tag{1.10}$$

which has been been proved by Shum in [19]. In particular, Alzer proved that if f is a positive decreasing function on $(0, T)$ satisfying (1.5) for all $t \in (0, T)$ then the function $f \in L^p(0, T]$ and

$$\left(\frac{1}{T}\int_0^T f^p(t)dt\right)^{1/p} \leq K_1 \left(\frac{1}{T}\int_0^T f^q(s)ds\right)^{1/q}, \tag{1.11}$$

holds with a new constant K_1 smaller than K and there exists a number $\delta > \varepsilon$ such that the inequality (1.11) holds not only for all $p \in [q, q+\varepsilon]$ but for all $p \in [q, q+\delta]$.

In recent years the study of dynamic inequalities on time scales has received a lot of attention, for more details we refer to the books [2, 3]. The general idea is to prove a result for an inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. This idea goes back to its founder Stefan Hilger [13] which started the study of dynamic equations on time scales. The study of dynamic inequalities on time scales helps avoid proving results twice - once for differential inequality and once again for difference inequality. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. For more details of time scale analysis we refer the reader to the two books by Bohner and Peterson [6], [7] which summarize and organize much of the time scale calculus.

Following this trend, we will prove new dynamic inequalities on time scales and use to prove some new integrability theorems. The rest of the paper is organized as follows: In Section 2, we recall some definitions and notations on time scales which will be used throughout the paper. In Section 3, first we prove a time scale version of Muckenhoupt's inequality, Bojarski, Sbordone and Wik inequality and Shum's inequality on time scales. Second, we apply Muckenhoupt's type inequality and Shum's type inequality on time scales to prove a new higher integrability results of Alzer's type on time scales. As special cases we will derive discrete versions from the obtained inequalities which are essentially new and the other different time scales will be left to the reader due to the limited space.

2. Preliminaries On Time scales

Let $\mathbb{T}$ be a time scale, which is an arbitrary nonempty closed subset of the real numbers. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. A time scale $\mathbb{T}$ equipped with the order topology is metrizable and is a K_σ−space; i.e. it is a union of at most countably many compact sets. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$.

The mapping $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, \infty)$ such that $\mu(t) := \sigma(t) - t$ is called the graininess. For any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. Define $f^\Delta(t)$ to be the number (if it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t with

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

In this case, we say $f^\Delta(t)$ is the (delta) derivative of f at t and that f is (delta) differentiable at t.

We will refer to the (delta) integral which we can define as follows. If $F^\Delta(t) = f(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t f(s)\Delta s := F(t) - F(a)$. If $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $F(t) := \int_{t_0}^t f(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $F^\Delta(t) = f(t)$, $t \in \mathbb{T}$. Note that if $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0$, and

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\mu(t) = 1$, and

$$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t).$$

If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\sigma(t) = t + h$, $\mu(t) = h$, and

$$\int_a^b f(t)\Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} f(a + kh)h.$$

If $\mathbb{T} = \{t : t = q^k,\ k \in \mathbb{N}_0,\ q > 1\}$, then $\sigma(t) = qt$, $\mu(t) = (q-1)t$, $\sigma(t) = qt$, $\mu(t) = (q-1)t$, and

$$\int_{t_0}^\infty f(t)\Delta t = \sum_{k=n_0}^\infty f(q^k)\mu(q^k), \quad \text{where } \ t_0 = q^{n_0}.$$

If $\mathbb{T}$ is an arbitrary time scale and the interval $[a, b) \subset \mathbb{T}$ contains only isolated points, then

$$\int_a^b f(t)\Delta t = \sum_{t \in [a,b)} (\sigma(t) - t)f(t).$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$, here $g^\sigma = g \circ \sigma$) of two differentiable functions f and g

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \text{ and } \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \tag{2.1}$$

The following simple consequence of Keller's chain rule (see [6, Theorem 1.90]) which is needed in the proof of the main results is given by

$$(u^\gamma(t))^\Delta = \gamma \int_0^1 [hu^\sigma + (1-h)u]^{\gamma-1}\, dhu^\Delta(t), \tag{2.2}$$

Without loss of generality, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[a, b]_\mathbb{T}$ by $[a, b]_\mathbb{T} := [a, b] \cap \mathbb{T}$. We say that $f : [0, T]_\mathbb{T} \to \mathbb{R}$ belongs to

$L^p_\Delta((0,T]_{\mathbb{T}})$ provided that either $\|f\|_p = \int_0^T |f|^p \Delta t < \infty$, if $1 < p < \infty$, or there exists a constant $C \in \mathbb{R}^+$ such that $\|f\|_\infty = |f| \leq C$, on $\mathbb{I}$ if $p = +\infty$. The integration by parts formula is given by

$$\int_a^b u(t)v^\Delta(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^\Delta(t)v^\sigma(t)\Delta t. \tag{2.3}$$

To prove the main results, we will use the following Hölder inequality [**?**, Theorem 6.13]: Let $a,\, b \in \mathbb{T}$. For $u,\, v \in C_{rd}(\mathbb{T},\, \mathbb{R})$, we have

$$\int_a^b |u(t)v(t)|\, \Delta t \leq \left[\int_a^b |u(t)|^q\, \Delta t\right]^{\frac{1}{q}} \left[\int_a^b |v(t)|^p\, \Delta t\right]^{\frac{1}{p}}, \tag{2.4}$$

where $p > 1$ and $1/p + 1/q = 1$.

Throughout this paper, we will assume (usually without mentioning) that the functions in the statements of the theorems are nonnegative and rd-continuous functions, $\Delta-$differentiable, locally delta integrable and the integrals considered are assumed to exist (finite i.e. convergent).

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