Reliability Estimation under Type-II Censored Data from The Generalized Bilal Distribution: comparison between Bayesian and classical approaches

Ayman M. Abd-Elrahman*

Department of Mathematics, Faculty of Science,
Assiut University, 71516 Assiut, Egypt.

Abstract

The main object of this article is the estimation of the unknown population parameters and reliability function for the generalized Bilal model under Type-II censored data. Both maximum likelihood and Bayesian estimates are considered. A Gibb's sampling procedure is used to draw Markov Chain Monte Carlo samples, which have been used to compute the Bayes estimates and also to construct their corresponding credible intervals with the help of two different importance sampling techniques. A simulation study is carried out to compare the accuracy of the resulting estimators. Application to a real data set is considered for the sake of illustration.

Keywords: Maximum likelihood estimation; Fisher information Matrix; Bayesian estimations; Gibb's sampling; Importance sampling techniques; Interval estimates.

Mathematics Subject Classifications: 62B15; 60E05; 62F10; 62N02; 62N05.

1 Introduction

The generalized Bilal (GB) model coincides with the distribution of the median in a sample of size three from the Weibull distribution. It was first introduced by Abd–Elrahman [1].

*E-mail address: <ayman275@aun.edu.eg>.

He showed that its failure rate function can be upside-down bathtub shaped. The failure rate can also be decreasing or increasing. Abd–Elrahman [1] presented a comparison among the GB and some other models. Where, following Vargo et al. [2], Glen [3], and Pasquale Erto [4], he plotted coefficient of variation (CV) against skewness (SK) for the various distribution models. The plot usually includes all possible pairs (CV, SK) that a model can attain. The set of values that the GB (CV, SK) pairs can assume fall in between the Weibull and the Log-normal models, helping to see some benefits from the GB distribution. Therefore, the GB model can be used for several practical data analysis.

Suppose that n items are put on a life–testing experiment and we observe only the first r failure times, say $x_1 < x_2 < \cdots < x_r$. Then, $\mathbf{x} = (x_1, x_2, \cdots, x_r)'$ is called a Type–II censored sample. The remaining $(n-r)$ items are censored and are only known to be grater than x_r. This article will be based on a Type–II censored sample drawn from the GB distribution. Type–II censoring have been discussed by too many authors, among them, Ahmad et al. [5], Raqab [6], Wu et al. [7], Chana et al. [8], ElShahat and Mahmoud [9] and Abd-Elrahman and Niazi [10].

Likewise the Weibull distribution, the CDF of the GB distribution can take any of the following functional forms:

$$\begin{aligned}
F_X(x;\, \beta,\, \lambda) &= 1 - e^{-2\,\beta\,x^{\lambda}}\left(3 - 2\,e^{-\,\beta\,x^{\lambda}}\right), \qquad x>0,\ (\beta, \lambda > 0), & (1)\\
F_X(x;\, \theta,\, \lambda) &= 1 - e^{-2\,(x/\theta)^{\lambda}}\left(3 - 2\,e^{-\,(x/\theta)^{\lambda}}\right), \quad x>0,\ (\theta, \lambda > 0), & (2)\\
F_X(x;\, \theta_1,\, \lambda) &= 1 - e^{-2\,x^{\lambda}/\theta_1}\left(3 - 2\,e^{-\,x^{\lambda}/\theta_1}\right), \quad x>0,\ (\theta_1, \lambda > 0),\\
F_X(x;\, \theta_2,\, \lambda) &= 1 - e^{-2\,(\theta_2\,x)^{\lambda}}\left(3 - 2\,e^{-\,(\theta_2\,x)^{\lambda}}\right), \quad x>0,\ (\theta_2, \lambda > 0).
\end{aligned}$$

It is well known that, based on the maximum likelihood method, the results of any statistical inference that may be obtained by using one of these forms is applied to the other functional forms. This is true by using some re–parametrization techniques together with the *Invariance property* of the maximum likelihood estimators, see *e. g.* Dekking et al. [11]. In this article, Formula (1) is used as the CDF of the GB distribution. The corresponding PDF and reliability function are, respectively, given by:

$$f_X(x;\, \beta,\, \lambda) = 6\,\beta\,\lambda\,x^{\lambda-1}e^{-2\,\beta\,x^{\lambda}}\left(1 - e^{-\,\beta\,x^{\lambda}}\right), \quad x>0,\ (\beta, \lambda > 0) \tag{3}$$

and

$$s(t) \;=\; e^{-2\,\beta\,t^{\lambda}}\left(3-2\,e^{-\,\beta\,t^{\lambda}}\right). \tag{4}$$

The qth quantile, x_q, is an important quantity, especially for generating random variates using the inverse transformation method. In view of (1), following Abd-Elrahman [12], x_q of the GB distribution is given by:

$$x_q = \left[\frac{1}{\beta}\,\ln\left(\frac{1}{\gamma(q)}\right)\right]^{1/\lambda}, \tag{5}$$

where

$$\gamma(q) = \begin{cases} 0.5+\sin(a_q+\pi/6) & \text{if } 0<q<0.5, \\ 0.5 & \text{if } q=0.5, \\ 0.5-\cos(a_q+\pi/3) & \text{if } 0.5<q<1, \end{cases}$$

for $a_q = \frac{1}{3}\,\arctan(\frac{2\sqrt{q(1-q)}}{2\,q-1})$.

The layout of this paper is organized as follows: In Section 2, the maximum likelihood (ML) estimates of the parameters β and λ as well as the reliability function $s(t)$ is discussed. By using the missing information principle, Variance–covariance matrix of the unknown population parameters is obtained, which have been used to construct the asymptotic confidence intervals for each of the unknown parameters and reliability function. In Section 3, two different importance sampling techniques are introduced. Each one have been used to compute the Bayes estimates of the parameters (β, λ) and the reliability function $s(t)$; and also to construct their corresponding credible intervals. In Section 4, in order to compare the proposed estimators, a simulation study has been performed. Further, in Section 5, for the sake of illustration, the results of Sections 2 and 3 are illustrated via a real data analysis. Finally, we draw some concluding remarks.

2 Maximum Likelihood Estimation

It follows from (1) and (3) that, based on a given Type-II censored sample $\mathbf{x}$ drawn from the GB distribution, the likelihood function of the parameters β and λ is given by:

$$L(\beta,\,\lambda|\mathbf{x}) \;\propto\; \beta^r\lambda^r\,e^{-2\,\beta\,T_1+T_2}, \tag{6}$$

where

$$T_1=(n-r)\,x_r^{\lambda}+\sum_{j=1}^{r} x_j^{\lambda},\quad T_2=(n-r)\,\ln(3-2\,e^{-\beta\,x_r^{\lambda}})+\lambda\sum_{j=1}^{r}\ln(x_j)+\sum_{j=1}^{r}\ln(1-e^{-\beta\,x_j^{\lambda}})\,.$$

2.1 When λ is known

In this case, for fixed λ, say $\lambda=\lambda^{(0)}$, let $\theta=1/\beta$ and $y_i=x_i^{\lambda^{(0)}}$, $i=1,\,2,\,\cdots r$. Then, $y_1,\cdots,y_r$ is a Type-II random sample from $Bilal(\theta)$ distribution. Abd-Elrahman and Niazi [10] established the existence and uniqueness theorem for the ML estimate (MLE) of the parameter θ, say $\hat{\theta}_M$. Again by using the invariance property of the ML method, the MLE for the parameter β is then by $\hat{\beta}_M(\lambda^{(0)})=1/\hat{\theta}_M$. Clearly, $\hat{\beta}_M(\lambda^{(0)})$ exists and it is unique.

In the following, we provide an iterative technique for finding $\hat{\beta}_M(\lambda^{(0)})$. In order to do this, let

$$W_1=\frac{\beta\,x_r^{\lambda^{(0)}}\,e^{-\beta\,x_r^{\lambda^{(0)}}}}{3-2\,e^{-\beta\,x_r^{\lambda^{(0)}}}},\qquad W_{2j}=\frac{\beta\,x_j^{\lambda^{(0)}}\,e^{-\beta\,x_j^{\lambda^{(0)}}}}{1-e^{-\beta\,x_j^{\lambda^{(0)}}}},\quad j=1,\,2,\,\cdots,\,r. \tag{7}$$

In view of (6) and (7), the likelihood equation of β is then given by:

$$\frac{\partial\ln L(\beta,\,\lambda^{(0)}|\mathbf{x})}{\partial\,\beta}\;=\;\frac{r+\,2\,(n-r)\,W_1+\sum_{j=1}^{r}W_{2j}}{\beta}-2\left((n-r)\,x_r^{\lambda^{(0)}}+\sum_{j=1}^{r}x_j^{\lambda^{(0)}}\right).$$

For $\nu=0,1,2,\cdots$, we calculate $\hat{\beta}_M(\lambda^{(0)})$ by using the following formula:

$$\hat{\beta}_M^{(\nu+1)}(\lambda^{(0)})=\left.\frac{r+2\;(n-r)\,W_1+\sum_{j=1}^{r}W_{2j}}{2\,\left((n-r)\,x_r^{\lambda}+\sum_{j=1}^{r}x_j^{\lambda}\right)}\right|_{\beta=\hat{\beta}_M^{(\nu)}(\lambda^{(0)}),\,\lambda=\lambda^{(0)}}, \tag{8}$$

iteratively until some level of accuracy is reached.

Remark 1 *Note that, all of the functions W_1 and W_{2j}, $j=1,2,\cdots,r$, which appear in (8), need to have some initial value for β, say $\hat{\beta}^{(0)}$. This initial value can be obtained based on the available Type–II censored sample as if it is complete, see Ng et al [13]. We use the moment estimator of β as a starting point in the iterations* (8). *That is, in view of (3), $\hat{\beta}^{(0)}$ is given by*

$$\hat{\beta}^{(0)} = \frac{5\, r}{6 \sum_{i=1}^{r} x_i^{\lambda^{(0)}}}. \tag{9}$$

2.2 When β is known

When β is assumed to be known, say $\beta^{(0)}$, it follows from (6) that, the likelihood equation of λ is given by

$$\frac{\partial \ln L(\beta^{(0)},\, \lambda|\mathbf{x})}{\partial\, \lambda} = \frac{r}{\lambda} - 2\,(n{-}r)\ln(x_r)\left(\beta^{(0)} x_r^{\lambda} - W_1\right) + \sum_{j=1}^{r} \ln(x_j)\left(1 - 2\,\beta^{(0)} x_j^{\lambda} + W_{2j}\right), \tag{10}$$

where W_1 and W_{2j}, $j=1,2,\cdots,r$, are as given by (7) after replacing β, $\lambda^{(0)}$ by $\beta^{(0)}$ and λ, respectively. In order to established the existence and uniqueness of the MLE for λ, the following theorem is needed.

Theorem 2.1 *For a given fixed value of the parameter $\beta=\beta^{(0)}$, the MLE for the parameter λ, $\hat{\lambda}_M(\beta^{(0)})$, exists and it is unique.*

Proof. See Appendix.

The MLE $\hat{\lambda}_M(\beta^{(0)})$ can be iteratively obtained by using Newton's method, *i. e.*,

$$\hat{\lambda}_M^{(\nu+1)}(\beta^{(0)}) = \hat{\lambda}_M^{(\nu)}(\beta^{(0)}) - \left\{ \frac{\lambda\, \mathcal{G}_1(\beta^{(0)},\, \lambda|\mathbf{x})}{\lambda\, \mathcal{G}_2(\beta^{(0)},\, \lambda|\mathbf{x}) + \mathcal{G}_1(\beta^{(0)},\, \lambda|\mathbf{x})} \right\}\Bigg|_{\lambda=\hat{\lambda}_M^{(\nu)}(\beta^{(0)})}, \tag{11}$$

for $\nu = 0, 1, 2, \cdots$, where $\mathcal{G}_1(\cdot,\, \lambda|\mathbf{x})$ is the first derivative of $\ln L(\cdot,\, \lambda|\mathbf{x})$ as given by (10); and $\mathcal{G}_2(\cdot,\, \lambda|\mathbf{x})$ is the second derivative, given in Appendix, with respect to (w.r.t.) λ.

Remark 2 *Again note that, we have only a Type–II censored sample, but the sample CV can be calculated based on this data as if it is complete. Equating the sample CV with its corresponding CV from the population would results in an equation of λ only. Its solution provides a good initial value for λ, $\hat{\lambda}_M^{(0)}$, that can be used as a starting point* (11). *This technique have been used by, e. g. Kundu and Howlader [14] and Abd-Elrahman [1].*

Here, the population CV of the GB distribution is given by

$$\mathcal{C}(\lambda) = \sqrt{\frac{(3^{m_2} - 2^{m_2})\,\Gamma\,(m_2)}{(3^{m_1} - 2^{m_1})^2\,(\Gamma\,(m_1))^2} - 1}, \qquad m_1 = 1 + \frac{1}{\lambda}, \quad m_2 = 1 + \frac{2}{\lambda}. \tag{12}$$

2.3 When both β and λ are unknown

In this case, first an initial value for λ, $\hat{\lambda}^{(0)}$, can be obtained as described in Section 2.2. Once $\hat{\lambda}^{(0)}$ is obtained, an initial value for the parameter β, $\hat{\beta}^{(0)}$, can be calculated as the right hand side of (9) after replacing $\lambda^{(0)}$ by $\hat{\lambda}^{(0)}$.

Based on the initials $\hat{\beta}^{(0)}$ and $\hat{\lambda}^{(0)}$, an updated value for β, $\hat{\beta}^{(1)}$, can be obtained by using (8). Similarly, based on the pair $(\hat{\beta}^{(1)}, \hat{\lambda}^{(0)})$, an updated value for λ, $\hat{\lambda}^{(1)}$, can be obtained by using (11), and so on. As a stopping rule, the iterations will be terminated after some value $s < 1000$ with a level of accuracy, $\epsilon \leq 1.2 \times 10^{-7}$, which is defined as

$$\epsilon = \left| \frac{\hat{\beta}^{(s+1)} - \hat{\beta}^{(s)}}{\hat{\beta}^{(s)}} \right| + \left| \frac{\hat{\lambda}^{(s+1)} - \hat{\lambda}^{(s)}}{\hat{\lambda}^{(s)}} \right| .$$

Hence, the limiting pair of estimates $(\hat{\beta}^{(s)}, \hat{\lambda}^{(s)})$ exists and it is unique, which would maximizes the likelihood function (6) w.r.t. the unknown population parameters β and λ. That is, $\hat{\beta}_M = \hat{\beta}^{(s)}$ and $\hat{\lambda}_M = \hat{\lambda}^{(s)}$. Substituting the values of β and λ in (4) by their MLEs, the MLE for reliability function $s(t)$ at some value of $t = t_0$ can then be obtained.

Fisher information matrix (FIM) can be used to compute asymptotic variances of the MLEs of the underlying population parameters. The following section concerns with obtaining the FIM about the two unknown parameters β and λ of the GB distribution whose CDF is given by (1) under Type–II censoring sample.

2.4 Asymptotic Variances and Covariance

Following Ng et al. [13], Abd-Elrahman [15] and Abd-Elrahman and Niazi [10], the well known *missing information principle* is used for obtaining the FIM of the GB distribution under Type–II censoring sample. In order to do this, first of all, suppose that, $\mathbf{x} = (x_1, x_2, \ldots, x_r)'$ and $\mathbf{Y} = (X_{r+1}, X_{r+2}, \ldots, X_n)'$ denote the ordered observed censored and the unobserved ordered data, respectively. The vector $\mathbf{Y}$ can be thought of as the missing data. Combine $\mathbf{x}$ and $\mathbf{Y}$ to form $\mathbf{W}$, which is the complete data set. Based on the data set $\mathbf{W}$, the amount of information data provide about the unknown parameters

θ and λ, which are involved in (2), is given by [Abd-Elrahman [1]]:

$$I^{\star}_{\mathbf{W}}(\theta,\,\lambda)=\;n\begin{bmatrix}\dfrac{1.924683\,\lambda^2}{\theta^2} & -\dfrac{0.056056}{\theta}\\ -\dfrac{0.056056}{\theta} & \dfrac{1.790613}{\lambda^2}\end{bmatrix}. \tag{13}$$

However, since:

1) In this article, we use Formula (1) as the CDF of the GB distribution. That is, the vector of parameters $(\theta,\,\lambda)'$ appears in (2) is transformed into $(\beta,\,\lambda)'$ with $\beta=\theta^{-\lambda}$.
2) This transformation is one-to-one and its inverse exists, i. e. $\theta=\beta^{-1/\lambda}$.

Then, $I_{\mathbf{W}}\,(\beta,\,\lambda)$ can be easily obtained to be

$$I_{\mathbf{W}}(\beta,\,\lambda)=A^T I^{\star}_{\mathbf{W}}\left(\beta^{-\frac{1}{\lambda}},\,\lambda\right)\,A,$$

where A and A^T are the transformation matrix (Jacobian) and its corresponding transpose, respectively, see Schervish [16]. Here, A is given by

$$A=\begin{bmatrix}-\left\{\lambda\,\beta^{(1+1/\lambda)}\right\}^{-1} & \ln(\beta)\left\{\lambda^2\,\beta^{1/\lambda}\right\}^{-1}\\ 0 & 1\end{bmatrix}$$

and, therefore, $I_{\mathbf{W}}\,(\beta,\,\lambda)$ is given by

$$I_{\mathbf{W}}\,(\beta,\,\lambda)=\begin{bmatrix}\dfrac{c_1}{\beta^2} & \dfrac{c_2-c_1\,\ln(\beta)}{\beta\,\lambda}\\ \dfrac{c_2-c_1\,\ln(\beta)}{\beta\,\lambda} & \dfrac{c_3+\ln(\beta)\,\{c_1\,\ln(\beta)-c_4\}}{\lambda^2}\end{bmatrix} \tag{14}$$

with $c_1{=}1.92468$, $c_2{=}0.05606$, $c_3{=}1.79061$ and $c_4{=}0.11211$.

Now, in order to obtain the expected ordered unobserved (missing) information matrix $I_{\mathbf{Y}}(\beta,\,\lambda)$, we use the theorem of Ng et al. [13]. The conditional distribution of each $X_s\in\mathbf{Y}$ given $X_s>x_r$ follows the truncated underlying distribution with left truncation at x_r, $s{=}r{+}1,\,r{+}2,\ldots,n$. Therefore, by using (1) and (3), we have

$$f(x|X_s>x_r;\,\beta,\,\lambda)=\frac{6\,\beta\,e^{-2\,\beta\,\left(x^{\lambda}-x_r^{\lambda}\right)}\,\left(1-e^{-\beta\,x^{\lambda}}\right)}{\left(3-2\,e^{-\beta\,x_r^{\lambda}}\right)},\quad x>x_r,\;(\beta,\,\lambda>0). \tag{15}$$

Hence, based on the conditional distribution (15), $I_{\mathbf{Y}}(\beta,\,\lambda)$ is then given by

$$I_{\mathbf{Y}|\mathbf{x}}(\beta,\,\lambda)=-(n-r)\,\mathbb{E}\begin{bmatrix}\dfrac{\partial^2\,\ln[f(x|X_s>x_r;\,\beta,\,\lambda)]}{\partial\,\beta^2} & \dfrac{\partial^2\,\ln[f(x|X_s>x_r;\,\beta,\,\lambda)]}{\partial\,\beta\,\partial\,\lambda}\\ \dfrac{\partial^2\,\ln[f(x|X_s>x_r;\,\beta,\,\lambda)]}{\partial\,\lambda\,\partial\,\beta} & \dfrac{\partial^2\,\ln[f(x|X_s>x_r;\,\beta,\,\lambda)]}{\partial\,\lambda^2}\end{bmatrix}. \tag{16}$$

In order to evaluate of the expectations involved in (16), calculations for the following expressions are required.

1) Part 1:-

$$I^{(k)}(y) = \int_y^{\infty} \{\ln(t)\}^k \, G_1(t)\, \mathrm{dt}, \qquad \mathrm{y} > 0, \quad \mathrm{k} = 0, 1, 2, \tag{17}$$

where

$$G_1(t) = \frac{t\, e^{-2\,t}\ \left[t\, e^{-t}\ +\ (1 -\ e^{-t})\ (2 - 3\, e^{-t})\right]}{1 - e^{-t}}.$$

Denote $I_0 = \lim_{\lambda\to 0^+} I^{(0)}(y) = 0.32078$, $I_1 = \lim_{\lambda\to 0^+} I^{(1)}(y) = 0.00934$ and $I_2 = \lim_{\lambda\to 0^+} I^{(2)}(y) = 0.13177$. Then, (17) can be written as

$$I^{(k)}(y) = I_k - \int_0^{y} \{\ln(t)\}^k \, G_1(t)\, \mathrm{dt}, \qquad \mathrm{y} > 0, \quad \mathrm{k} = 0, 1, 2. \tag{18}$$

The integrals involved in (18) can be calculated by using a simple numerical integration tool, *e. g.* Simpson's rule.

2) Part 2:-

$$\begin{aligned} I^{(4)}(y) &= \int_y^{\infty} \frac{t^2\, e^{-3\,t}}{1 - e^{-t}}\, \mathrm{dt} \ = \ \mathrm{I}_4 - \int_0^{\mathrm{y}} \frac{\mathrm{t}^2\, \mathrm{e}^{-3\,\mathrm{t}}}{1 - \mathrm{e}^{-\mathrm{t}}}\, \mathrm{dt}, \qquad \mathrm{y} > 0, \\ &= I_4 - \sum_{j=0}^{\infty} \left\{ \int_0^{y} t^2\, e^{-(j+3)\,t}\, \mathrm{dt} \right\}, \\ &= e^{-3\,y} \sum_{j=0}^{\infty} \frac{\left(1 + (1 + (3+j)\,y)^2\right) e^{-j\,y}}{(3+j)^3}, \end{aligned} \tag{19}$$

where $I_4 = \lim_{y\to 0^+} I^{(4)}(y) = -\frac{9}{4} + 2\, \sum_{i=1}^{\infty} i^{-3} = 0.154114$.

Now, in view of (18) and (19), it is easy to show that the elements $I_{i\,j}$ of $I_{\mathbf{Y}|\mathbf{x}}(\beta,\, \lambda)$ after division by $(n-r)$, $i,\ j = 1, 2$, are given by

$$I_{11} \ = \ \frac{1}{\beta^2} \left\{ 1 + 6\, \left(\frac{e^{-y} I^{(4)}(y)}{3 - 2\, e^{-y}} - \frac{y^2 e^{-y}}{(3 - 2\, e^{-y})^2} \right) \right\}, \quad y = \beta\, x_r^{\lambda}, \tag{20}$$

$$I_{12} \ = \ -\frac{6}{\beta\,\lambda} \left\{ \frac{t_1(x_r) + \left[I^{(0)}(y) - \ln(\beta)\, I^{(1)}(y)\right]\ e^{2\,y}}{(3 - 2\, e^{-y})} \right\} = I_{21}, \tag{21}$$

$$I_{22} \ = \ \frac{1}{\lambda^2} \left\{ 1 + \frac{6\left[e^{2\,y}\left[(\ln(\beta))^2 I^{(0)}(y) - 2\ln(\beta)\, I^{(1)}(y) + I^{(2)}(y)\right] - t_2(x_r)\right]}{(3 - 2\, e^{-y})} \right\}, \tag{22}$$

where

$$t_1(x_r)=\frac{\beta\, x_r^{\lambda}\ln\left(x_r^{\lambda}\right)\left[\left(1-e^{-\beta\, x_r^{\lambda}}\right)\left(3-2\, e^{-\beta\, x_r^{\lambda}}\right)+\beta\, x_r^{\lambda}e^{-\beta\, x_r^{\lambda}}\right]}{\left(3-2\, e^{-\beta\, x_r^{\lambda}}\right)}$$

and

$$t_2(x_r)=\frac{\beta\, x_r^{\lambda}\left(\ln\left(x_r^{\lambda}\right)\right)^2\left[\beta\, x_r^{\lambda}e^{-\beta\, x_r^{\lambda}}+\left(1-e^{-\beta\, x_r^{\lambda}}\right)\left(3-2\, e^{-\beta\, x_r^{\lambda}}\right)\right]}{\left(3-2\, e^{-\beta\, x_r^{\lambda}}\right)}.$$

Note that, the elements I_{ij}, $i, j = 1, 2$, constitute the Fisher information related to each X_s, $s = r+1, r+2, \cdots, n$, where X_s is distributed as in (15). Therefore, in view of (20–22), the elements of the FIM about the parameters β and λ related to the complete data set $\mathbf{W}$ can be obtained as: $n\, \lim_{y\to 0^+} I_{ij}$, i, $j = 1, 2$, which give as the same results as in (14).

Therefore, the FIM gains about the two unknown parameters β and λ from a given Type–II censored sample, $(x_1, x_2, \cdots x_r)'$, from the GB distribution whose CDF is as given by (1), is then given by

$$I_{\mathbf{x}}(\beta,\,\lambda) = I_{\mathbf{W}}(\beta,\,\lambda) - I_{\mathbf{Y}|\mathbf{x}}(\beta,\,\lambda).$$

Once $I_{\mathbf{x}}(\beta,\,\lambda)$ is calculated, at $\beta = \hat{\beta}_M$ and $\lambda = \hat{\lambda}_M$, the asymptotic variance–covariance matrix of the MLEs of the two unknown parameters β and λ is then given by

$$\mathbf{Var} - \mathbf{Cov}\left(\hat{\beta}_M,\,\hat{\lambda}_M\right) = I_{\mathbf{x}}^{-1}\left(\hat{\beta}_M,\,\hat{\lambda}_M\right) = \begin{bmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} \\ \hat{\sigma}_{21} & \hat{\sigma}_2^2 \end{bmatrix}.$$

Again, once $I_{\mathbf{x}}^{-1}\left(\hat{\beta}_M,\,\hat{\lambda}_M\right)$ is obtained, the asymptotic variance of the reliability function $s(t_0)$ can then be calculated as the lower bound of the Cramér–Rao inequality of the variance of any unbiased estimator for $s(t_0)$. That is,

$$\begin{aligned}\mathrm{Var}[\widehat{s(t_0)}] &= \left\{\left[\begin{array}{cc}\frac{\partial s(t_0)}{\partial \beta} & \frac{\partial s(t_0)}{\partial \lambda}\end{array}\right] I_{\mathbf{x}}^{-1}\left(\beta_M,\,\lambda_M\right)\left[\begin{array}{c}\frac{\partial s(t_0)}{\partial \beta}\\ \frac{\partial s(t_0)}{\partial \lambda}\end{array}\right]\right\}\Bigg|_{\beta=\hat{\beta}_M,\,\lambda=\hat{\lambda}_M}, \\ &= 36\, t_0^{2\,\hat{\lambda}_M} e^{-4\,\hat{\beta}_M\, t_0^{\hat{\lambda}_M}}\left[\hat{\sigma}_2^2\,\hat{\beta}_M^2\left(\ln(t_0)\right)^2 + \hat{\beta}_M\,\ln(t_0)\,\hat{\sigma}_{12} + \hat{\sigma}_1^2\right]\left(1-e^{-\hat{\beta}_M\, t_0^{\hat{\lambda}_M}}\right)^2.\end{aligned}$$

Consequently, the asymptotic $(1-\alpha)\,100\,\%$ confidence intervals, ACIs, for $\hat{\beta}_M$, $\hat{\lambda}_M$ and $\widehat{s(t_0)}_M$ are given by

$$\left[\hat{\beta}_M \mp Z_{\frac{\alpha}{2}}\,\hat{\sigma}_1\right], \qquad \left[\hat{\lambda}_M \mp Z_{\frac{\alpha}{2}}\,\hat{\sigma}_2\right] \qquad \text{and} \qquad \left[\widehat{s(t_0)}_M \mp Z_{\frac{\alpha}{2}}\sqrt{\mathrm{Var}[\widehat{s(t_0)}]}\right], \tag{23}$$

respectively, where $Z_{\frac{\alpha}{2}}$ is the percentile $(1-\frac{\alpha}{2})$ of the standard normal distribution.

3 Bayesian Estimation

In this section, Bayesian estimators for the two unknown population parameters and reliability function are obtained. Their associated $(1-\alpha)100\,\%$ highest posterior density (HPD) credible intervals, are also obtained. Although we have discussed mainly the squared error loss (SEL) function, any other loss function can easily be considered.

It is assumed that β and λ have two independent gamma priors with the hyper parameters $a_1 > 0$ and $b_1 > 0$ for β; and $a_2 > 0$ and $b_2 > 0$ for λ. That is,

$$\pi_1(\beta) \propto \beta^{a_1-1}e^{-b_1\beta} \qquad \text{and} \qquad \pi_2(\lambda) \propto \lambda^{a_2-1}e^{-b_2\lambda}. \tag{24}$$

Moreover, Jeffrey's priors can be obtained as special cases of (24) by substituting $a_1 = b_1 = a_2 = b_2 = 0$. The hyper parameters can be chosen to suit the prior belief of the experimenter in terms of location and variability of the prior distribution. Combining (6) and (24), the joint posterior density function of β and λ is then given by

$$\pi(\beta,\, \lambda|\mathbf{x}) \propto \; \beta^{r+a_1-1}e^{-(b_1+2\,T_1)\,\beta}\lambda^{r+a_2-1}e^{-b_2\lambda}e^{T_2}, \tag{25}$$

where T_1 and T_2 are as given in (6). The Bayes estimate of any function $g(\beta,\, \lambda)$, under a SEL function, is given by

$$\widehat{g(\beta,\, \lambda)}_B = \frac{\int_0^\infty\int_0^\infty \, g(\beta,\, \lambda)\;\pi(\beta,\, \lambda|\mathbf{x})\,\mathrm{d}\beta\,\mathrm{d}\lambda}{\int_0^\infty\int_0^\infty \, \pi(\beta,\, \lambda|\mathbf{x})\,\mathrm{d}\beta\,\mathrm{d}\lambda}. \tag{26}$$

The integrals involved in (26) are usually not obtainable in closed form, but Lindley's approximation [17] may be used to compute such ratio of integrals. It cannot however be used to construct credible intervals. Therefore, following Kundu and Howlader [14], we approximate (26) by using a Gibb's sampling procedure to draw Markov Chain Monte Carlo samples, which can be used to compute the Bayes estimates and also to construct their corresponding HPD credible intervals. We propose two different importance sampling techniques, which will be denoted as IS1 and IS2. The corresponding credible intervals can then be constructed as suggested by Chen and Shao [18].

3.1 First importance sampling technique (IS1)

The joint posterior density function (25) can be rewritten as

$$\pi(\beta,\, \lambda|\mathbf{x}) \propto \pi_1^\star(\beta|\lambda,\, \mathbf{x})\, \pi_2^\star(\lambda|\mathbf{x})\, h_3(\beta,\, \lambda), \tag{27}$$

where $\pi_1^\star(\beta|\lambda,\, \mathbf{x})$ is a gamma density function given by

$$\pi_1^\star(\beta|\lambda,\, \mathbf{x}) \propto \beta^{r+a_1-1} e^{-(b_1+2\,T_1)\,\beta}, \tag{28}$$

$\pi_2^\star(\lambda|\mathbf{x})$ is a proper density function given by

$$\pi_2^\star(\lambda|\mathbf{x}) \propto \frac{\lambda^{r+a_2-1} e^{-b_2\lambda} \prod_{j=1}^{r} x_j^\lambda}{(b_1 + 2\,T_1)^{r+a_1}} \tag{29}$$

and

$$h_3(\beta,\, \lambda) = \left(1 - \frac{2}{3}\, e^{-\beta\, X_r^\lambda}\right)^{n-r} \prod_{j=1}^{r} \left(1 - e^{-\beta\, X_j^\lambda}\right). \tag{30}$$

Now, since $\pi_1^\star(\beta|\lambda,\, \mathbf{x})$ follows a gamma distribution then, it is quite simple to generate from it. On the other hand, although the function $\pi_2^\star(\lambda|\mathbf{x})$ is a proper density, we can use the method developed by Devroye [19] for generating λ. This method requires to ensure that (29) has a log-concave density function property. Therefore, the following theorem is needed.

Theorem 3.1 *The function $\pi_2^\star(\lambda|\mathbf{x})$, given by (29), has a log–concave density function.*

Proof. See Appendix.

Using Theorem 3.1, a simulation based consistent estimate of $g(\beta,\, \lambda)$ can be obtained by using the following algorithm.

Algorithm 1.

Step 1: Generate λ from $\pi_2^{\star}(\cdot|\mathbf{x})$, by using the method developed by Devroye [19].

Step 2: Generate β from $\pi_1^{\star}(\cdot|\lambda,\, \mathbf{x})$.

Step 3: Repeat Step 1 and Step 2 to obtain (β_i, λ_i), $i = 1,\, 2,\, \cdots,\, M$.

Step 4: For $i = 1,\, 2,\, \cdots,\, M$, calculate g_i as $g(\beta_i,\, \lambda_i)$; and ω_i as $\dfrac{h_3(\beta_i,\, \lambda_i)}{\sum_{i=1}^{M} h_3(\beta_i,\, \lambda_i)}$, where $h_3(\beta,\, \lambda)$ is as given by (30).

Step 5: Under a SEL function, an approximate Bayes estimate of $g(\beta,\, \lambda)$ and its corresponding estimated variance can be, respectively, obtained as

$$\hat{g}(\beta,\, \lambda)_{_{IS1}} = \sum_{i=1}^{M} \omega_i\, g_i \quad \text{and} \quad \hat{V}[g(\beta,\, \lambda)]_{_{IS1}} = \sum_{i=1}^{M} \omega_i \left(g_i - \hat{g}(\beta,\, \lambda)_{_{IS1}}\right)^2. \tag{31}$$

3.2 Second importance sampling technique (IS2)

In this technique, we will start with another rewriting to the joint posterior density function (25) as

$$\pi(\beta,\, \lambda|\mathbf{x}) \propto \pi_1^{\star}(\beta|\lambda,\, \mathbf{x})\, \pi_3^{\star}(\lambda|\mathbf{x})\, h_4(\beta,\, \lambda), \tag{32}$$

where $\pi_1^{\star}(\beta|\lambda,\, \mathbf{x})$ is as given by (28), while $\pi_3^{\star}(\lambda|\mathbf{x})$ is a gamma density function given by

$$\pi_3^{\star}(\lambda|\mathbf{x}) \propto \lambda^{r+a_2-1} \exp\left[-\left(b_2 + \sum_{j=1}^{r-1} \ln\left(\frac{x_r}{x_j}\right)\right)\lambda\right]. \tag{33}$$

This is true, since $b_2 > 0$ and $\frac{x_r}{x_j} > 1$, $j = 1,\, 2,\, \cdots r-1$. Therefore,

$$h_4(\beta,\, \lambda) = \frac{x_r^{r\,\lambda}\left(1 - \frac{2}{3}\, e^{-\beta\, X_r^{\lambda}}\right)^{n-r} \prod_{j=1}^{r}\left(1 - e^{-\beta\, X_j^{\lambda}}\right)}{(b_1 + 2\, T_1)^{r+a_1}}. \tag{34}$$

In this technique, since $\pi_1^{\star}(\beta|\lambda,\, \mathbf{x})$ and $\pi_3^{\star}(\lambda,\, \mathbf{x})$ follow a gamma distribution each, it is quite simple to generate from them. Therefore, it is straight forward that a simulation based consistent estimate of $g(\beta, \lambda)$ can be obtained using the following algorithm:

Algorithm 2.

Step 1: Generate $\lambda^{\star}$ from $\pi_3^{\star}(\cdot|\mathbf{x})$.

Step 2: Generate $\beta^{\star}$ from $\pi_1^{\star}(\cdot|\lambda^{\star},\, \mathbf{x})$.

Step 3: Repeat Step 1 and Step 2 to obtain $(\beta_i^{\star}, \lambda_i^{\star})$, $i = 1,\, 2,\, \cdots,\, M$.

Step 4: For $i = 1, 2, \cdots, M$, calculate $g_i^\star$ as $g(\beta_i^\star, \lambda_i^\star)$; and $\omega_i^\star$ as $\dfrac{h_4(\beta_i^\star, \lambda_i^\star)}{\sum_{i=1}^{M} h_4(\beta_i^\star, \lambda_i^\star)}$, where $h_4(\beta, \lambda)$ is as given by (34).

Step 5: In this case, based on a SEL function, the approximate Bayes estimate of $g(\beta, \lambda)$ and its corresponding estimated variance can be, respectively, obtained as

$$\hat{g}(\beta, \lambda)_{_{IS2}} = \sum_{i=1}^{M} \omega_i^\star\, g_i^\star \quad \text{and} \quad \hat{V}[g(\beta, \lambda)]_{_{IS2}} = \sum_{i=1}^{M} \omega_i^\star \left(g_i^\star - \hat{g}(\beta, \lambda)_{_{IS2}}\right)^2. \tag{35}$$

By using the idea of Chen and Shao [18], based on (g_i, ω_i) (or $(g_i^\star, \omega_i^\star)$), $i = 1, 2, \cdots, M$, the $(1-\alpha)\,100\,\%$ HPD credible interval of $g(\beta, \lambda)$ related to IS1 (or IS2) technique can be easily obtained.

4 Simulation Study

This section is devoted to compare the performance of the proposed Bayes estimators with the MLEs, we carry out a simulation study using different sample sizes (n), different effective sample sizes (r), and for different priors (non-informative and informative). For prior information we have used: Non-informative prior, Prior 1 with $a_1 = b_1 = a_2 = b_2 = 0$, and informative prior, Prior 2 with $a_1 = 2$, $b_1 = 4$, $a_2 = 3$, $b_2 = 4$. The IMSL [20] routines *DRNUN* and *DRNGAM* are used in the generation of the uniform and gamma random variates, respectively. In computing the estimates, first we generate β and λ from gamma (a_1, b_1) and gamma (a_2, b_2) distributions, respectively. These generated values are $\beta_0 = 0.5439$ and and $\lambda_0 = 0.7468$. The corresponding value of the reliability function calculated at $t_0 = 0.9$ is 0.8299 . Second, we generate 5000 samples from the GB distribution with $\beta = 0.5439$ and $\lambda = 0.7468$. For the importance sampling techniques (IS1 and IS2), we set $M = 15,000$, when we apply Algorithm 1 or 2. The average estimate of $\vartheta^\star$ and the associated mean squared error (MSEs) are computed, respectively, as:

$$\text{Average} = \frac{1}{5000} \sum_{k=1}^{5000} \vartheta^\star, \qquad \text{MSE} = \frac{1}{5000} \sum_{k=1}^{5000} \left(\vartheta_k^\star - \vartheta\right)^2,$$

where $\vartheta^\star$ stands for an estimator (ML or Bayes) of β, λ or $s(0.9)$, at the k^{th} iteration; and ϑ stands for $\beta_0 = 0.5439$, $\lambda_0 = 0.7468$, or $s(0.9) = 0.8299$. The computational results are displayed in Tables 1–3, where the first entry in each cell is for the average estimate and the second entry, which is given in parentheses, is for the corresponding MSE. It has been noticed from Tables 1–3, that

1) As expected, the MSEs of all estimates (ML or Bayes) decrease as n or r increases.

2) The Bayes estimators under Prior 1 or Prior 2 by using IS2 technique are mainly better than the corresponding estimators by using IS1 technique in terms of in terms of average bias and MSE.

3) In all cases, the MSEs of the MLEs are less than the corresponding Bayes estimators under Prior 1 by using IS1 technique. On the other hand, the performances in terms of average bias and the MSE of the Bayes estimators under Prior 1 by using IS2 technique and the MLE are very similar.

4) For small and moderate sample or censoring sizes, the Bayes estimators under Prior 2 by using IS2 technique clearly outperform the MLEs in terms of average bias and MSE.

5) For large sample or censoring sizes, the performances in terms of average bias and the MSE of the Bayes estimators under Prior 2 with IS2 technique and the MLE are very similar.

5 DataAnalysis

This section concerns with illustration of the methods presented in Sections 2 and 3, where a real data set is considered. This data set is from Hinkley [21] and consists of thirty successive values of March precipitation in Minneapolis/St Paul. This data is used by Barreto-Souza and Cribari-Neto [22] in fitting the generalized exponential-Poisson distribution (GEP), and by Abd-Elrahman [1, 12] in fitting the Bilal and GB distributions. For the complete sample case, the MLEs of β and λ, respectively, are 2.0156 and 1.2486, which are obtained as described in Section 2. The negative of the Log Likelihood, Kolmogorov-Smirnov (K-S) statistics and its corresponding p-value related to these MLEs are 38.1763, 0.0532 and 1.0, respectively. These results agree with the results in Abd-Elrahman [1]. Based on this p-value, it is clear that the GB distribution is found to fit the data very well.

If only the first 20 data points are observed. The corresponding sample mean and CV of this 20 observed sample points are 1.1225 and 0.4206, respectively. Equating the right hand side of (12) by 0.4206 and solving for λ would results in the unique solution $\lambda_0 = 1.7385$. Based on this value of λ, it follows from (9) that β_0 is calculated as 0.6147. The iterative scheme, which is described in Section 2, starts with the initials $\lambda_{(0)} = 1.7385$ and $\beta_{(0)} = 0.6147$. The estimates of β and λ, converge to $\hat{\beta}_M = 0.41417$ and $\hat{\lambda}_M = 1.29926$ with a level of accuracy less than 1.2×10^{-10} of the absolute relative errors. From these data, we have

$$I_{\mathbf{W}}(\hat{\beta}_M, \hat{\lambda}_M) = \begin{pmatrix} 336.60039 & 97.70695 \\ 97.70695 & 60.15509 \end{pmatrix},$$

and

$$I_{\mathbf{Y}}(\hat{\beta}_M, \hat{\lambda}_M) = \begin{pmatrix} 81.73227 & 53.50402 \\ 53.50402 & 35.95699 \end{pmatrix}.$$

Hence,

$$I_{\mathbf{x}}(\hat{\beta}_M, \hat{\lambda}_M) = \begin{pmatrix} 254.86812 & 44.20293 \\ 44.20293 & 24.19810 \end{pmatrix}.$$

Therefore, the estimated variance–covariance matrix of $\hat{\beta}_M$ and $\hat{\lambda}_M$ is

$$I_{\mathbf{x}}^{-1}(\hat{\beta}_M, \hat{\lambda}_M) = \begin{pmatrix} 0.00574 & -0.01049 \\ -0.01049 & 0.06049 \end{pmatrix}.$$

Therefore, the standard errors of the MLEs of β and λ are 0.07576 and 0.24595, respectively. The MLE of $s(0.9)$ and its corresponding asymptotic standard error are 0.78002 and 0.06340, respectively. The 99 % ACIs for β, λ and $s(0.9)$ are (0.21897, 0.60938), (0.66575, 1.93278) and (0.61672, 0.94331), respectively.

On the other hand, the simulation study given in Section 4 shows that, the Bayes estimators by using IS2 technique is better than the corresponding estimators obtained by using IS1 technique in terms of average bias and MSE. Therefore, under non-informative prior, we compute Bayes estimate by generating an importance sample of size $M = 15,000$ with their corresponding importance weights according to Algorithm 2. The Bayes estimates of β, λ and $s(0.9)$, and their corresponding standard errors (given in parentheses), respectively, are $\hat{\beta}_{IS2} = 0.39034\,(0.04907)$, $\hat{\lambda}_{IS2} = 1.34910\,(0.19207)$ and $\widehat{s(0.9)}_{IS2} = 0.79899\,(0.03866)$. The 99 % HPD credible intervals for β, λ and $s(0.9)$ are (0.24320, 0.43781), (0.85632, 1.92996) and (0.73657, 0.91060), respectively.

6 Concluding Remarks

(1) In this article, the ML and Bayes estimation of the parameters as well as the reliability function of the GB distribution based on a given Type-II censored sample are obtained.

(2) The existence and uniqueness theorem for the ML estimator of the population parameter λ, when β is assumed to be known, is established. An iterative procedure for finding the ML estimators of the two unknown population parameters, is also provided. The elements of the FIM are obtained, and they have been used in turn for calculating the asymptotic confidence intervals of λ, β and the reliability function.

(3) Two different importance sampling techniques have been proposed, which can be used for further Bayesian studies.

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The author hereby confirms that this manuscript is in its original form and it is not considered for publication elsewhere.

Appendix

Proof of Theorem 2.1

It follows from (10), that the second of $\ln L(\beta,\, \lambda|\mathbf{x})$ is given by

$$\mathcal{G}_2(\beta,\, \lambda|\mathbf{x}) = -\frac{r}{\lambda^2} - \frac{6\,(n-r) z\, f_1(z)\ (\ln(x_r))^2}{(3\, e^z - 2)^2} - \sum_{j=1}^{r} \frac{y_j\, f_2(y_j)\ (\ln(x_j))^2}{(e^{y_j}-1)^2}, \tag{A1}$$

where $z = \beta\, x_r^\lambda,\ f_1(z) = e^z\, [z + e^z\, (1 - e^{-z})\, (3 - 2\, e^{-z})],\ y_j = \beta\, x_j^\lambda,\ j = 1,\, 2,\, \cdots,\, r,$ and $f_2(y_j) = 2\, e^{2\, y_j} - 5\, e^{y_j} + 3 + y_j\, e^{y_j}$.

Now, in order to prove that $\mathcal{G}_2(\beta,\, \lambda|\mathbf{x}) < 0$, it is sufficient to show that $f_1(z) > 0$ and $f_2(y_j) > 0$. It is clear that $f_1(z) > 0$. On the other hand, by expanding the exponential functions involved in $f_2(y_j)$ about $z = 0$, $f_2(y_j)$ can be rewritten as

$$f_2(y_j) \ = \ y_j^2 + \sum_{k=2}^{\infty} \frac{y_j^k\, \left(2^{k+1} - 5 + y_j\right)}{k!} > 0.$$

Therefore, $\dfrac{\partial^2 \ln L(\beta,\, \lambda|\mathbf{x})}{\partial \lambda^2} < 0$. This implies that the ML estimate, $\hat{\lambda}_M$, for λ is unique.

To insure that $\hat{\lambda}_M$ exists, following Balakrishnan et al. [23], we rewrite (10) as $h_1(\lambda) = h_2(\lambda)$, where $h_1(\lambda) = r/\lambda$ and

$$h_2(\lambda) = -2\,(n-r)\ln(x_r)\left(\beta\, x_r^\lambda - W_1\right) + \sum_{j=1}^{r} \ln(x_j)\left(1 + W_{2j} - 2\,\beta\, x_j^\lambda\right),$$

where W_1 and W_{2j}, $j = 1,\, 2,\, \cdots,\, r$, are as given in (10).

Note that,

$$\begin{aligned} \ell_1 \ &= \ \lim_{\lambda \to 0^+} h_2(\lambda) = 2\ (n-r)\,[\beta - \eta_1(\beta)]\ln(x_r) - \sum_{j=1}^{r} \ln(x_j)\,[1 - 2\,\beta + \eta_2(\beta)]\,, \\ \ell_2 \ &= \ \lim_{\lambda \to \infty} h_2(\lambda) = \left(\ell_\infty + \sum_{i=1}^{r} \ell_{2i}\right) > 0, \end{aligned}$$

where $\eta_1(\beta) = \frac{\beta}{3\, e^\beta - 2}$, $\eta_2(\beta) = \frac{\beta}{e^\beta - 1}$,

$$\ell_\infty = \begin{cases} 0 & \text{if } 0 < x_r \le 1, \\ \infty & \text{if } x_r > 1. \end{cases}, \qquad \ell_{2i} = \begin{cases} 2\, \ln\left(\frac{1}{x_i}\right) & \text{if } 0 < x_i \le 1, \\ \infty & \text{if } x_i > 1. \end{cases}.$$

Furthermore, it follows from (A1), that

$$\frac{\partial h_2(\lambda)}{\partial \lambda} = \frac{6\,(n-r)\beta\, x_r^\lambda f_1\left(\beta\, x_r^\lambda\right)\ (\ln(x_r))^2}{\left(3\, e^{\beta\, x_r^\lambda} - 2\right)^2} + \sum_{j=1}^{r} \frac{\beta\, x_j^\lambda f_2\left(\beta\, x_j^\lambda\right)\ (\ln(x_j))^2}{\left(e^{\beta\, x_j^\lambda} - 1\right)^2} > 0,$$

which implies that $\ell_1 < \ell_2$. Therefore, $h_2(\lambda)$ is an increasing function of λ. But $h_1(\lambda)$ is a positive strictly decreasing function with right limit $+\infty$ at 0. This insures that $h_1(\lambda) = h_2(\lambda)$ holds exactly once at some value $\lambda = \lambda^{\diamond}$. Hence, the theorem is proved.

Proof of Theorem 3.1

It follows from (29) that, the logarithm base e of $\pi_2^{\star}(\lambda|\mathbf{x})$ without the additive constant is given by

$$\ln\{\pi_2^{\star}(\lambda|\mathbf{x})\} = (r + a_2 - 1)\ln(\lambda) + \left(\sum_{j=1}^{r}\ln(X_j) - b_2\right)\lambda - \ln\left(\frac{b_1}{2} + T_1\right)(r + a_1).$$

Therefore,

$$\frac{\mathrm{d}^2\ln\{\pi_2^{\star}(\lambda|\mathbf{x})\}}{\mathrm{d}\lambda^2} = -\frac{r+a_2-1}{\lambda^2} - (r + a_1)\frac{\partial^2\ln\{\xi(\lambda)\}}{\partial\,\lambda^2},$$

where $\xi(\lambda) = \frac{b_1}{2} + (n-r)\,x_r^{\lambda} + \sum_{j=1}^{r} x_j^{\lambda}$. In order to show that $\frac{\mathrm{d}^2\ln\{\pi_2^{\star}(\lambda|\mathbf{x})\}}{\mathrm{d}\lambda^2} < 0$, it is sufficient to show that $\xi_1 = \xi''(\lambda)\,\xi(\lambda) - \{\xi'(\lambda)\}^2 > 0$. This is true, because

$$\begin{aligned}
\xi_1 &= \left(\frac{b_1}{2} + (n-r)\,x_r^{\lambda} + \sum_{j=1}^{r} x_j^{\lambda}\right)\left((n-r)\,x_r^{\lambda}(\ln(x_r))^2 + \sum_{j=1}^{r} x_j^{\lambda}(\ln(x_j))^2\right) \\
&\quad - \left((n-r)\,x_r^{\lambda}\ln(x_r) + \sum_{j=1}^{r} x_j^{\lambda}\ln(x_j)\right)^2, \\
&= \frac{b_1}{2}\left((n-r)\,x_r^{\lambda}(\ln(x_r))^2 + \sum_{j=1}^{r} x_j^{\lambda}(\ln(x_j))^2\right) + (n-r)\,x_r^{\lambda}\sum_{j=1}^{r} x_j^{\lambda}\left(\ln\left(\frac{x_j}{x_r}\right)\right)^2 \\
&\quad + \sum_{j=1}^{r}\left(\sum_{k=1}^{r} a(j,k)\right), \qquad a(j,k) = \ln(x_k)\,x_k^{\lambda}\,x_j^{\lambda}(\ln(x_k) - \ln(x_j)).
\end{aligned}$$

Since $\sum_{j=1}^{r}\left(\sum_{k=1}^{r} a(j,k)\right) = \left(\sum_{j=1}^{r} a(j,j)\right) + \left(\sum_{j=1}^{r}\sum_{k=j+1}^{r}(a(j,k) + a(k,j))\right)$. Then,

$$\begin{aligned}
\xi_1 &= \frac{b_1}{2}\left((n-r)\,x_r^{\lambda}(\ln(x_r))^2 + \sum_{j=1}^{r} x_j^{\lambda}(\ln(x_j))^2\right) + (n-r)\,x_r^{\lambda}\sum_{j=1}^{r} x_j^{\lambda}\left(\ln\left(\frac{x_j}{x_r}\right)\right)^2 \\
&\quad + \sum_{j=1}^{r}\left(\sum_{k=j+1}^{r} x_k^{\lambda}\,x_j^{\lambda}(\ln(x_k) - \ln(x_j))^2\right) > 0.
\end{aligned}$$

Hence, the theorem is proved.

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Table 1: Average estimates of β and the associated MSEs.

n	r	MLE	Bayes Prior 1		Bayes Prior 2	
			IS1	IS2	IS1	IS2
25	15	0.5535 (0.0134)	0.5189 (0.0144)	0.5381 (0.0129)	0.5274 (0.0109)	0.5381 (0.0101)
	20	0.5432 (0.0104)	0.5118 (0.0122)	0.5411 (0.0099)	0.5216 (0.0097)	0.5413 (0.0083)
	25	0.5405 (0.0096)	0.5121 (0.0115)	0.5478 (0.0091)	0.5200 (0.0092)	0.5456 (0.0077)
30	20	0.5476 (0.0093)	0.4971 (0.0119)	0.5256 (0.0097)	0.5096 (0.0093)	0.5291 (0.0080)
	25	0.5427 (0.0083)	0.4945 (0.0112)	0.5362 (0.0082)	0.5072 (0.0088)	0.5362 (0.0072)
	30	0.5412 (0.0079)	0.4955 (0.0108)	0.5412 (0.0075)	0.5069 (0.0087)	0.5397 (0.0067)
40	30	0.5447 (0.0060)	0.4647 (0.0125)	0.5117 (0.0071)	0.4784 (0.0098)	0.5149 (0.0063)
	35	0.5428 (0.0057)	0.4647 (0.0121)	0.5236 (0.0060)	0.4764 (0.0097)	0.5250 (0.0055)
	40	0.5421 (0.0056)	0.4656 (0.0116)	0.5294 (0.0056)	0.4775 (0.0095)	0.5279 (0.0051)

Table 2: Average estimates of λ and the associated MSEs.

n	r	MLE	Bayes Prior 1		Bayes Prior 2	
			IS1	IS2	IS1	IS2
25	15	0.8355 (0.0499)	0.8570 (0.0597)	0.8167 (0.0465)	0.8264 (0.0363)	0.8006 (0.0300)
	20	0.8049 (0.0274)	0.8248 (0.0339)	0.7794 (0.0236)	0.8063 (0.0234)	0.7736 (0.0180)
	25	0.7889 (0.0172)	0.8056 (0.0216)	0.7431 (0.0147)	0.7943 (0.0159)	0.7456 (0.0122)
30	20	0.8095 (0.0306)	0.8477 (0.0437)	0.7976 (0.0298)	0.8223 (0.0291)	0.7875 (0.0222)
	25	0.7928 (0.0204)	0.8278 (0.0290)	0.7707 (0.0183)	0.8093 (0.0210)	0.7684 (0.0148)
	30	0.7817 (0.0136)	0.8123 (0.0201)	0.7306 (0.0128)	0.7995 (0.0153)	0.7344 (0.0109)
40	30	0.7857 (0.0165)	0.8543 (0.0335)	0.7774 (0.0174)	0.8318 (0.0242)	0.7738 (0.0143)
	35	0.7782 (0.0128)	0.8400 (0.0257)	0.7588 (0.0125)	0.8248 (0.0197)	0.7589 (0.0107)
	40	0.7720 (0.0094)	0.8272 (0.0185)	0.7036 (0.0117)	0.8151 (0.0151)	0.7089 (0.0102)

Table 3: Average estimates of $s(0.9)$ and the associated MSEs.

n	r	MLE	Bayes Prior 1		Bayes Prior 2	
			IS1	IS2	IS1	IS2
25	15	0.8284 (0.0079)	0.8570 (0.0088)	0.8403 (0.0075)	0.8487 (0.0067)	0.8393 (0.0060)
	20	0.8344 (0.0067)	0.8601 (0.0080)	0.8355 (0.0062)	0.8517 (0.0063)	0.8350 (0.0052)
	25	0.8357 (0.0064)	0.8590 (0.0077)	0.8287 (0.0058)	0.8523 (0.0061)	0.8304 (0.0049)
30	20	0.8310 (0.0059)	0.8727 (0.0080)	0.8482 (0.0062)	0.8618 (0.0062)	0.8449 (0.0051)
	25	0.8339 (0.0055)	0.8736 (0.0077)	0.8385 (0.0053)	0.8629 (0.0060)	0.8384 (0.0046)
	30	0.8346 (0.0053)	0.8722 (0.0075)	0.8328 (0.0049)	0.8627 (0.0059)	0.8342 (0.0044)
40	30	0.8318 (0.0040)	0.8985 (0.0089)	0.8577 (0.0048)	0.8866 (0.0069)	0.8550 (0.0043)
	35	0.8329 (0.0038)	0.8978 (0.0086)	0.8474 (0.0041)	0.8879 (0.0068)	0.8464 (0.0037)
	40	0.8331 (0.0038)	0.8966 (0.0082)	0.8405 (0.0038)	0.8866 (0.0067)	0.8419 (0.0034)