LINEAR ONE FORM DEFORMATION OF SPRAYS

S. G. ELGENDI

Abstract. In this paper, the linear one form deformation of a flat spray is investigated. The metrizability of the deformation spray is characterized. New projectively flat Reimannain metrics are obtained. These new metrics are not, generally, isometric to the Klein metric via affine transformations. New Finsler solutions for Hilbert's fourth problem are constructed. Various examples are studied.

1. Introduction

The notion of sprays was introduced by W. Ambrose et al. [1] in 1960. A system of second order ordinary differential equations (SODE) with positively 2-homogeneous coefficients functions can be shown as a second order vector field, which is called a spray. All sprays are associated with a SODE and conversely, a spray can be associated with a SODE. If such a system introduces the variational (Euler-Lagrange) equations of the energy of a Finsler metric, then it is said to be Finsler metrizable and in this case the spray is the geodesic spray of the Finsler metric. The Finsler metrizability problem for a spray S looking for a Finsler structure whose geodesics coincide with the geodesics of S. The metrizability problem can be considered as a special case of the inverse problem of the calculus of variation. Several interesting results on the metrizability problem can be found in the literature, we refer, for example, to [5, 9, 12, 13, 16] and the references therein.

The geodesics of a Finsler structure F on an open subset $U \subset \mathbb{R}^n$ are straight lines if and only if the spray coefficients of F are given in the form $G^i = P(x,y)y^i$. Straight lines in U are parametrized by $\sigma(t) = f(t)a + b$, where $a, b \in \mathbb{R}^n$ are constant vectors and $f(t) > 0$ is a positive function. The regular case of Hilbert's Fourth Problem is to characterize all locally projectively flat Finsler metrics; that is, the metrics whose geodesics are straight lines on an open subset of $\mathbb{R}^n$. Beltrami's theorem states that a Riemannian metric is locally projectively flat if and only if it has constant sectional curvature. In Finslerian case, this is not true. There are non projectively flat Finsler metrics of constant flag curvature. Flag curvature is an analogue of sectional curvature in Finsler geometry.

Bucataru and Muzany ([6], [7]) characterized the sprays which are metrizable by Finsler metrics of constant flag curvature κ.

In this paper, we introduce the linear one form deformation. For simplicity, we consider the linear one form deformation of a flat spray; namely, $S = S_0 - 2\beta C$ and $\beta(x,y) = y^k b_k(x)$ is a liner one form on the manifold M. We study the Finsler metrizability of the deformation spray S. We characterize the metrizability of S by a Finsler metric of constant flag curvature. We obtain a new class of projectively flat metrics of constant flag curvature and hence new solutions for Hilbert's fourth problem.

The Klein metric on $\mathbb{B}^n \subset \mathbb{R}^n$ is given by

$$F = \sqrt{\frac{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}{(1-|x|^2)^2}}, \qquad y \in T_x\mathbb{B}^n \simeq \mathbb{R}^n.$$

The Klein metric F is projectively flat Riemannian metric with projective factor $P = \frac{\langle x,y\rangle}{1-|x|^2}$. It is known that every locally projectively flat Riemannian metric is locally isometric to F. In this paper, we obtained new class of projectivley flat metrics of constant flag curvature. This class is

2010 *Mathematics Subject Classification.* 53C60, 53B40, 58B20, 49N45, 58E30.
Key words and phrases. Sprays, metrizability, projective deformation, Hilbert's fourth problem, Klein metrics, projectively flat, isometry.

given by

$$F = \sqrt{\frac{4h(x)c_{ij}y^iy^j - 4(c_{ij}x^iy^j)^2 - 4\langle c', y\rangle c_{ij}x^iy^j - \langle c', y\rangle^2}{(2(h(x)))^2}}$$

and its projective factor is

$$P(x,y) = -\frac{2c_{ij}x^iy^j + \langle c', y\rangle}{2(h(x))},$$

where $h(x) := c_{ij}x^ix^j + \langle c', x\rangle + c$, $c_{ij} = c_{ji}$, c, $c' = (c_1, c_2, ..., c_n)$ are constants. Starting by the Klein metric, the general linear transformation, x to $Ax + B$ and y to Ay where A is an $n \times n$ invertible matrix and B is an arbitrary $n \times 1$ matrix, generates projectively flat Riemannian metrics. The obtained class of projectively flat metrics is not, generally, isometric to Klein metric via affine transformations.

Since the deformation spray S of a flat spray S_0 is always isotropic and in the case that the curvature of S is non zero, then the metric freedom [10] of S is unique up to some constants. Hence, in our case the deformation of a flat spray by the specific one form is metrizable by unique metric. However, we construct new projectively flat Finsler metrics and hence Finsler solutions for Hilbert's fourth problem. Also, these solutions are new solutions for the system [3], [14]

$$(\Phi_\pm)_{x^k} = \Phi(\Phi_\pm)_{y^k}, \ \Phi_\pm = P \pm \sqrt{-\kappa}F.$$

It is known that, [6], one of the conditions for a spray S with non-vanishing Ricci curvature to be metrizable by a Finsler function of non-zero constant flag curvature is rank $dd_J(\text{Tr }\Phi) = 2n$. As an application of the deformation of a flat spray by a linear one form, we answer the following question:

Does any spray of non-vanishing Ricci curvature satisfy the condition rank $dd_J(Tr\ \Phi) = 2n$?

By an example, we show that for a spray S, if S has non vanishing Ricci curvature, then the rank of the form $dd_J(\text{Tr }\Phi)$ is not necessarily maximal; that is, the condition rank $dd_J(\text{Tr }\Phi) = 2n$ is sharp for the metrizability of S.

2. Preliminaries

Let M be an n-dimensional manifold and (TM, π_M, M) be its tangent bundle and $(\mathcal{T}M, \pi, M)$ the subbundle of nonzero tangent vectors. We denote by (x^i) local coordinates on the base manifold M and by (x^i, y^i) the induced coordinates on TM. The vector 1-form J on TM defined, locally, by $J = \frac{\partial}{\partial y^i} \otimes dx^i$ is called the natural almost-tangent structure of TM. The vertical vector field $\mathcal{C} = y^i \frac{\partial}{\partial y^i}$ on TM is called the canonical or the Liouville vector field.

A vector field $S \in \mathfrak{X}(\mathcal{T}M)$ is called a spray if $JS = \mathcal{C}$ and $[\mathcal{C}, S] = S$. Locally, a spray can be expressed as follows

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}, \tag{2.1}$$

where the *spray coefficients* $G^i = G^i(x, y)$ are 2-homogeneous functions in the $y = (y^1, \ldots, y^n)$ variable. A curve $\sigma : I \to M$ is called regular if $\sigma' : I \to \mathcal{T}M$, where σ' is the tangent lift of σ. A regular curve σ on M is called *geodesic* of a spray S if $S \circ \sigma' = \sigma''$. Locally, $\sigma(t) = (x^i(t))$ is a geodesic of S if and only if it satisfies the equation

$$\frac{d^2x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0. \tag{2.2}$$

An orientation preserving reparameterization $t \to \tilde{t}(t)$ of the system (2.2) leads to a new spray $\widetilde{S} = S - 2PC$. The scalar function $P \in C^\infty(\mathcal{T}M)$ is 1-homogeneous and it is related to the new parameter by

$$\frac{d^2\tilde{t}}{dt^2} = P\Big(x^i(t), \frac{dx^i}{dt}\Big)\frac{d\tilde{t}}{dt}, \ \frac{d\tilde{t}}{dt} > 0. \tag{2.3}$$

Definition 2.1. Two sprays S and $\widetilde{S}$ are projectively related if their geodesics coincide up to an orientation preserving reparameterization. $\widetilde{S}$ is called the projective deformation of spray S.

A nonlinear connection is defined by an n-dimensional distribution $H : u \in \mathcal{T}M \to H_u \in T_u(\mathcal{T}M)$ that is supplementary to the vertical distribution, which means that for all $u \in \mathcal{T}M$, we have $T_u(\mathcal{T}M) = H_u(\mathcal{T}M) \oplus V_u(\mathcal{T}M)$.

Every spray S induces a canonical nonlinear connection through the corresponding horizontal and vertical projectors,

$$h = \frac{1}{2}(Id + [J,S]), \quad v = \frac{1}{2}(Id - [J,S]) \tag{2.4}$$

Equivalently, the canonical nonlinear connection induced by a spray can be expressed in terms of an almost product structure $\Gamma = [J,S] = h - v$. With respect to the induced nonlinear connection, a spray S is horizontal, which means that $S = hS$. Locally, the two projectors h and v can be expressed as follows

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \qquad v = \frac{\partial}{\partial y^i} \otimes \delta y^i,$$

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j}, \quad \delta y^i = dy^i + N_i^j(x,y)dx^i, \quad N_i^j(x,y) = \frac{\partial G^j}{\partial y^i}.$$

The Jacobi endomorphism is defined by

$$\Phi = v \circ [S,h] = R_j^i \frac{\partial}{\partial y^i} \otimes dx^j = \left(2\frac{\partial G^i}{\partial x^j} - S(N_j^i) - N_k^i N_j^k\right)\frac{\partial}{\partial y^i} \otimes dx^j.$$

The two curvature tensors are related by

$$3R = [J,\Phi], \quad \Phi = i_S R.$$

The Ricci curvature, Ric, and the Ricci scalar, $\rho \in C^\infty(\mathcal{T}M)$ [2] and [15], are given by

$$\mathrm{Ric} = (n-1)\rho = R_i^i = \mathrm{Tr}(\Phi).$$

Definition 2.2. A spray S is called *isotropic* if the Jacobi endomorphism has the form

$$\Phi = \rho J - \alpha \otimes C,$$

where α is a semi-basic 1-form $\alpha \in \Lambda^1(\mathcal{T}M)$.

Due to the homogeneity condition, for isotropic sprays, the Ricci scalar is given by $\rho = i_S\alpha$.

Definition 2.3. A Finsler manifold of dimension n is a pair (M,F), where M is a differentiable manifold of dimension n and F is a map

$$F : TM \longrightarrow \mathbb{R},$$

such that*:*

(a): F is smooth and strictly positive on $\mathcal{T}M$ and $F(x,y) = 0$ if and only if $y = 0$,
(b): F is positively homogenous of degree 1 in the directional argument y*:* $\mathcal{L}_\mathcal{C} F = F$,
(c): The metric tensor $g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j}$ has rank n on $\mathcal{T}M$, where $E := \frac{1}{2}F^2$ is the energy function.

Since the 2-form $dd_J E$ is non-degenerate, the Euler-Lagrange equation

$$\omega_E := i_S dd_J E - d(E - \mathcal{L}_C E) = 0 \tag{2.5}$$

uniquely determines a spray S on TM. This spray is called the *geodesic spray* of the Finsler function. The ω_E is called the Euler-Lagrange form associated to S and E.

Definition 2.4. A spray S on a manifold M is called *Finsler metrizable* if there exists a Finsler function F such that the geodesic spray of the Finsler manifold (M,F) is S.

Definition 2.5. The function F is said to be of scalar flag curvature if there exists a function $\kappa \in C^\infty(\mathcal{T}M)$ such that

$$\Phi = \kappa(F^2 J - F d_J F \otimes C).$$

It follows that for a Finsler function F, of scalar flag curvature κ, its geodesic spray S is isotropic, with Ricci scalar $\rho = \kappa F^2$ and the semi-basic 1-form $\alpha = \kappa F d_J F$.

Definition 2.6. A Finsler metric $F = F(x,y)$ on an open subset $U \subset \mathbb{R}^n$ is said to be projectively flat if all geodesies are straight lines in U. A Finsler metric F on a manifold M is said to be locally projectively flat if at any point, there is a local coordinate system (x^i) in which F is projectively flat.

From now on, we use the notations ∂_i for the partial differentiation with respect to x^i and $\dot{\partial}_i$ for the partial differentiation with respect to y^i.

By [11], a Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is projectively flat if and only if it satisfies the following system of equations,

$$y^j \dot{\partial}_i \partial_j F - \partial_i F = 0, \tag{2.6}$$

In this case, $G^i = Py^i$ where $P = P(x, y)$, the projective factor of F, given by $P = \frac{\partial_k F y^k}{2F}$.

3. Linear one form deformation

In this section, we consider the deformation spray $S = S_0 - 2\beta C$, where S_0 is a flat spray; that is, $S_0 = y^i \partial_i$ and $\beta = b_i(x) y^i$ is one form on M. By [4], one has the following

Lemma 3.1. *For the deformation spray $S = S_0 - 2\beta C$ of a flat spray S_0. The corresponding horizontal projectors and Jacobi endomorphisms of the two sprays are related as follows:*

(a): $h = h_0 - \beta J - d_J \beta C$,

(b): $\Phi = (\beta^2 - S_0 \beta) J - (\beta d_J \beta + d_J(S_0 \beta) - 3 d_{h_0} \beta) \otimes C$,

Proposition 3.2. *Let $S = S_0 - 2\beta C$ be a linear one form deformation of a flat spray S_0 with non-vanishing Ricci curvature. Then, the properties $d_J \alpha = 0$ and rank $dd_J(Tr\ \Phi) = 2n$ hold if and only if*

(a): $\partial_i b_j - \partial_j b_i = 0$, *i.e b_i is gradient (β is closed on M), and*

(b): $\det(\partial_i b_j + b_i b_j) \neq 0$.

Proof. Let $d_J \alpha = 0$, since $d_J \alpha = -3 d_{h_0} d_J \beta$, then we have

$$d_{h_0} d_J \beta(\partial_i, \partial_j) = \partial_i \dot{\partial}_j \beta - \partial_j \dot{\partial}_i \beta = 0.$$

Using the fact $\dot{\partial}_j \beta = b_j$, we get $\partial_i b_j - \partial_j b_i = 0$, i.e b_i is gradient.

Since Tr $\Phi = (n-1)(\beta^2 - S_0 \beta)$, then by a direct calculations and using that b_i is gradient, we obtain

$$dd_J(\text{Tr}\ \Phi) = 2(n-1)((\partial_i b_j + b_i b_j) dx^i \wedge dy^j + (b_i \partial_j \beta - b_j \partial_i \beta) dx^i \wedge dx^j).$$

Consequently, rank $dd_J(\text{Tr}\ \Phi) = 2n$ if $\det(\partial_i b_j + b_i b_j) \neq 0$.

Conversely, if (a) and (b) are satisfied then the result follows. □

The above proposition together with [6] show the following

Corollary 3.3. *Let $S = S_0 - 2\beta C$ be a linear one form deformation of a flat spray S_0 with non-vanishing Ricci curvature. Then, the following properties:*

(P1): *S is Finsler metrizable,*

(P2): *S is metrizable by a Finsler metric of non-vanishing scalar flag curvature,*

(P3): *S is Finsler metrizable by an Einstein metric,*

(P4): *S is metrizable by a Finsler metric of non-zero constant flag curvature,*

(P5): *S is Ricci constant,*

are equivalent if and only if

(a): $\partial_i b_j - \partial_j b_i = 0$, *i.e b_i is gradient.*

(b): $\det(\partial_i b_j + b_i b_j) \neq 0$.

Now, we introduce one of the main results of this work.

Theorem 3.4. *The linear one form deformation $S = S_0 - 2\beta C$, $\beta(x, y) = y^k b_k(x)$, of a flat spray S_0, is metrizable by a Finsler function of constant flag curvature $\kappa \neq 0$ if and only if*

$$b_k(x) = -\frac{2c_{ik}x^i + c_k}{2(c_{ij}x^i x^j + \langle c', x \rangle + c)}, \tag{3.1}$$

where $c_{ij} = c_{ji}$, c, $c' = (c_1, c_2, ..., c_n)$ are constants.

Proof. By [7], one can see that the spray $S = S_0 - 2\beta C$ is metrizable by a Finsler function of non zero constant flag curvature if and only if

C1): $d_J \alpha = 0$, α is a semi-basic 1-form given by $\alpha = \beta d_J \beta + d_J(S_0 \beta) - 3 d_{h_0} \beta$,

C2): $d_h \rho = 0$, ρ is the Ricci scalar given by $\rho = \beta^2 - S_0 \beta$,

C3): $\mathrm{rank}(\mathrm{dd}_J\rho) = 2\mathrm{n}$.

Now, consider the deformation $S = S_0 - 2\beta C$, where $\beta = y^k b_k$,

$$b_k = -\frac{2c_{ik}x^i + c_k}{2(c_{ij}x^i x^j + \langle c', x\rangle + c)}.$$

Since $d_J\alpha = -3d_{h_0}d_J\beta$, then we have

$$d_{h_0}d_J\beta(\partial_i, \partial_j) = \partial_i\dot{\partial}_j\beta - \partial_j\dot{\partial}_i\beta = \partial_i b_j - \partial_j b_i.$$

Using the property that c_{ij} is symmetric, b_i is gradient and therefore $d_J\alpha = 0$.

To calculate ρ, let's compute $S_0(\beta) = y^k\partial_k\beta$,

$$S_0(\beta) = -\frac{2h(x)c_{ij}y^i y^j - \big(2c_{ij}x^i y^j + \langle c', y\rangle\big)^2}{2(h(x))^2}, \tag{3.2}$$

for simplicity, we use $h(x) := c_{ij}x^i x^j + \langle c', x\rangle + c$. Using the formula of β together with (3.2), we have

$$\rho = \frac{4h(x)c_{ij}y^i y^j - 4(c_{ij}x^i y^j)^2 - 4\langle c', y\rangle c_{ij}x^i y^j - \langle c', y\rangle^2}{(2h(x))^2}. \tag{3.3}$$

Differentiating (3.3) with respect to ∂_k and $\dot{\partial}_k$, we obtain:

$$\begin{aligned} \partial_k\rho \;=\; & \frac{4\big(2c_{ik}x^i + c_k\big)\Big(4\big(c_{ij}x^i y^j\big)^2 + 4\langle c', y\rangle c_{ij}x^i y^j + \langle c', y\rangle^2\Big)}{(2h(x))^3} \\ & - \frac{4c_{ij}y^i y^j\big(2c_{ik}x^i + c_k\big) + 8c_{ij}x^i y^j c_{ik}y^i + 4\langle c', y\rangle c_{ik}y^i}{(2h(x))^2}, \end{aligned} \tag{3.4}$$

$$\dot{\partial}_k\rho \;=\; \frac{8h(x)c_{ik}y^i - 8c_{ij}x^i y^j c_{kr}x^r - 4c_k c_{ij}x^i y^j - 4\langle c', y\rangle c_{kj}x^j - 2\langle c', y\rangle c_k}{(2h(x))^2}. \tag{3.5}$$

Now, substituting from (3.3), (3.4) and (3.5) into $d_h\rho = d_{h_0}\rho - \beta d_J\rho - 2\rho d_J\beta$, we get $d_h\rho = 0$.

Putting $\rho_{ij} := \dot{\partial}_i\dot{\partial}_j\rho$, then we get

$$\rho_{ij} = \frac{4c_{ij}h(x) - 4(c_{ik}x^k)(c_{jk}x^k) - 2c_i c_{jk}x^k - 2c_j c_{ik}x^k - c_i c_j}{(2h(x))^2}.$$

The condition **C3)** (regularity condition) is satisfied if $\det(\rho_{ij}) \neq 0$. Consequently, for appropriate constants c_{ij}, c_i and c such that $\det(\rho_{ij}) \neq 0$, we have a projectively flat Finsler metric of constant flag curvature.

Conversely, let S be metrizable. Since the condition **C1)** is satisfied if and only if there exists a locally defined, 0-homogeneous, smooth function g on $\Omega \times \mathbb{R}^n\backslash\{0\}$, Ω is open subset of $\mathbb{R}^n$, such that

$$d_J\beta = d_{h_0}g.$$

Locally, we have

$$d_J\beta(\partial_i) = d_{h_0}g(\partial_i) \Rightarrow \dot{\partial}_i(b_j y^j) = \partial_i g \Rightarrow b_i = \partial_i g.$$

Since b_i is a function of x, then $g(x, y) = g_1(x) + g_2(y)$, $g_2(y)$ is 0-homogenous function. Then, we can write $\beta = y^i b_i(x) = S_0(g)$ and $b_i(x) = \partial_i g$. The condition **C2)** is satisfied if and only if

$$d_{h_0}\rho - S_0(g)d_J\rho - 2\rho d_{h_0}g = 0.$$

Applying the above equation on ∂_i and using that $S_0(h) = \beta$ and $\rho = \beta^2 - S_0(\beta)$, we have

$$\partial_i\rho - \beta\dot{\partial}_i\rho - 2\rho\partial_i g = 0. \tag{3.6}$$

Making use of $\beta = y^i\partial_i g$ and $\rho = \beta^2 - S_0\beta$, the function

$$g(x, y) = -\frac{1}{2}\ln\big(c_{ij}x^i x^j + \langle c', x\rangle + c\big) + g_2(y), \tag{3.7}$$

is a solution of (3.6). Differentiating (3.7) with respect to ∂_k we have

$$b_k(x) = \partial_k g = -\frac{2c_{ik}x^i + c_k}{2(c_{ij}x^i x^j + \langle c', x\rangle + c)}. \tag{3.8}$$

□

Since the relation between the Ricci scalar ρ and the Finsler function F of constant flag curvature κ is given by $\rho = \kappa F^2$, then have the following

Proposition 3.5. *With appropriate constants c_{ij}, c, $c' = (c_1, c_2, ..., c_n)$, the class*

$$F = \sqrt{\frac{4h(x)c_{ij}y^iy^j - 4(c_{ij}x^iy^j)^2 - 4\langle c', y\rangle c_{ij}x^iy^j - \langle c', y\rangle^2}{(2h(x))^2}} \tag{3.9}$$

introduces projectively flat metrics of non zero constant flag curvature. Therefore, we have new solutions for Hilbert's fourth problem.

As a special case, we have the following

Corollary 3.6. *Taking $c_{ij} = \mu\delta_{ij}$, $c_i = 0$, $c = 1$, then we get*

$$P = -\frac{\mu\langle x, y\rangle}{1+\mu|x|^2}, \quad F_\mu = \sqrt{\frac{(1+\mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}{(1+\mu|x|^2)^2}}, \quad \kappa = \mu, \quad F_\mu^2 = \rho/\mu$$

and g_{ij} is given by

$$g_{ij} = \frac{\delta_{ij}}{1+\mu|x|^2} - \frac{\mu x_i x_j}{(1+\mu|x|^2)^2}.$$

When $\mu = -1$, F_{-1} is the well known Klein metric on the standard unit ball $\mathbb{B}^n \subset \mathbb{R}^n$.

Corollary 3.7. *Let $S = S_0 - 2\beta C$ be a deformation of a flat spray S_0,*

$$\beta = -\frac{2c_{ij}x^iy^j + \langle c', y\rangle}{2(c_{ij}x^ix^j + \langle c', x\rangle + c)}.$$

*A necessary condition for the properties **(P1)-(P5)** to be equivalent is $c_{ij} \neq 0$.*

Remark 3.8. The deformation of a flat spray by a linear one form gives the following advantages:
– It is a way to obtain projectively flat Riemannian metrics of non zero constant flag curvature,
– It gives new solutions to the system [3], [14]

$$(\Phi_\pm)_{x^k} = \Phi(\Phi_\pm)_{y^k}, \ \Phi_\pm = P \pm \sqrt{-\kappa}F,$$

– It gives new solutions for Hilbert's fourth problem.

It is known that, [6], one of the conditions for a spray S with non-vanishing Ricci curvature to be metrizable by a Finsler function of non-zero constant flag curvature is rank $dd_J(\text{Tr }\Phi) = 2n$. As an application of the deformation of a flat spray by a linear one form, we answer the following question:

Does any spray of non-vanishing Ricci curvature satisfy the condition rank $dd_J(\text{Tr }\Phi) = 2n$?

The following proposition shows that, for a spray S, if S has non vanishing Ricci curvature, then the rank of the form $dd_J(\text{Tr }\Phi)$ not necessarily maximal; that is, the condition rank $dd_J(\text{Tr }\Phi) = 2n$ is sharp for the metrizability of S.

Proposition 3.9. *Let Φ the Jacobi endomorphism of a spray S , then we have:*
(a): *If rank $dd_J(\text{Tr }\Phi) = 2n$, then S has non-vanishing Ricci curvature.*
(b): *If S has non-vanishing Ricci curvature, then rank $dd_J(\text{Tr }\Phi)$ is not necessarily maximal.*

Proof. The proof of **(a)** is obvious, so we prove **(b)** only. The proof of **(b)** can be performed by providing an example in which S has non-vanishing Ricci curvature and $dd_J(\text{Tr }\Phi)$ has not maximal rank. Let

$$S = S_0 - 2\beta C, \quad \beta = -\frac{\langle c', y\rangle}{2(\langle c', x\rangle + c)}, \quad b_i(x) = -\frac{c_i}{2(\langle c', x\rangle + c)},$$

where $c' = (c_1, c_2, ..., c_n)$, c_i and c are arbitrary constants. Since $\text{Tr }\Phi = \text{Ric} = (n-1)(\beta^2 - S_0\beta)$, Ric is the Ricci curvature. Then, we get

$$\text{Ric} = (1-n)\frac{\langle c', y\rangle^2}{4(\langle c', x\rangle + c)^2}.$$

Since $n \neq 1$, we have $\text{Ric} \neq 0$. Straightforward calculations lead to

$$dd_J(\text{Tr }\Phi) = 2(n-1)(\alpha_{ij}dx^i \wedge dy^j + \beta_{ij}dx^i \wedge dx^j),$$

where $\alpha_{ij} = \frac{c_ic_j}{4(\langle c', x\rangle + c)^2}$. Then, $dd_J(\text{Tr }\Phi)$ has maximal rank if $\det(\alpha_{ij}) \neq 0$, but $\det(\alpha_{ij}) = 0$ and moreover $\text{rank}(\alpha_{ij}) = 1$, then the result follows. □

4. Affine transformations of Klien metric

The Klein metric on $\mathbb{B}^n \subset \mathbb{R}^n$ is given by

$$F_{-1} = \sqrt{\frac{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}{(1-|x|^2)^2}}, \quad P = \frac{\langle x, y\rangle}{1-|x|^2}, \qquad y \in T_x\mathbb{B}^n \simeq \mathbb{R}^n.$$

The Klein metric F_{-1} is projectively flat Riemannian metrics. It is known that every locally projectively flat Riemannian metric is locally isometric to F_{-1}. Starting by the Klein metric, the affine transformation, x to $Ax + B$ and y to Ay where A is an $n \times n$ invertible matrix and B is an arbitrary $n \times 1$ matrix, produces projectively flat Riemannian metrics.

Theorem 4.1. *The class* (3.9) *is not, generally, isometric to the Klein metric via affine transformations.*

Proof. Consider the affine transformation x to $Ax + B$ and y to Ay, where A is an $n \times n$ invertible matrix and B is an arbitrary $n \times 1$ matrix,

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

The Klein metric transforms to

(4.1)
$$F = \sqrt{\frac{H(x)(\sum_{k=1}^n a_{ki}a_{kj})y^iy^j + ((\sum_{k=1}^n a_{ki}a_{kj})x^iy^j)^2 + 2\langle B', y\rangle(\sum_{k=1}^n a_{ki}a_{kj})x^iy^j + \langle B', y\rangle^2}{(H(x))^2}},$$

where $H(x) := 1-(\sum_{k=1}^n a_{ki}a_{kj})x^ix^j - 2(\sum_{k=1}^n a_{ki}b_k)x^i - |B|^2$, $B' = (B_1, ..., B_n)$, $B_i = \sum_{k=1}^n a_{ki}b_k$. Now, comparing the equations (3.9) and (4.1), we get

$$2c_{ij} = -\sum_{k=1}^n a_{ki}a_{kj}, \qquad c_i = -\sum_{k=1}^n a_{ki}b_k, \qquad 2c = 1 - |B|^2.$$

Thus we get, formally, the class (3.9). So once you have the transformation then you obtain the $c's$, but if you have the $c's$ then the transformation not necessarily exist.

For example, let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Then the constants $c's$ are given by

$$\begin{aligned} c_{11} &= -\frac{1}{2}(a_{11}^2 + a_{21}^2) \\ c_{22} &= -\frac{1}{2}(a_{12}^2 + a_{22}^2) \\ c_{12} = c_{21} &= -\frac{1}{2}(a_{11}a_{12} + a_{21}a_{22}) \\ c_1 &= -(a_{11}b_1 + a_{21}b_2) \\ c_2 &= -(a_{12}b_1 + a_{22}b_2) \\ c &= \frac{1}{2}(1 - b_1^2 - b_2^2). \end{aligned}$$

So if the matrices A and B are given, then one can get the $c's$ but if the $c's$ are given, generally, the above system is inconsistent. For instance, take $c_{11} = c_{22} = 0, c_{12} = c_{21} = 1, c_1 = 1, c_2 = 1, c = 1$, we get a projectively flat metric and at the same time by substitution in the above system one obtains inconsistent system. For this choice of the $c's$, we have

$$F = \sqrt{\frac{(8x_1x_2y_1y_2 - 4x_1^2y_2^2 - 4x_2^2y_1^2 + 4x_1y_1y_2 - 4x_1y_2^2 - 4x_2y_1^2 + 4x_2y_1y_2 - y_1^2 + 6y_1y_2 - y_2^2)}{4(2x_1x_2 + x_1 + x_2 + 1)^2}},$$

and the projective factor is given by

$$\beta = -\frac{1}{2}\frac{2x_1y_2 + 2x_2y_1 + y_1 + y_2}{2x_1x_2 + x_1 + x_2 + 1}.$$

So one can say that the affine transformation of Klien metric is contained in (3.9) but not any metric in (3.9) can be obtained by an affine transformation. □

Now, the question is

What is the isometry (transformation) between the klein metric and the class (3.9)*?*

5. Finsler solutions for Hilbert fourth problem and examples

Since the deformation spray S of a flat spray S_0 is always isotropic and in the case which the curvature of S is non zero, then the metric freedom [10] of S is unique up to some constants. Therefore, in our case the deformation of a flat spray by the specific linear one form $\beta = b_i(x)y^i$ where $b_i(x)$ given by (3.1) is metrizable by unique Riemannain metric given in (3.9). However, in this section, we introduce some new projectively flat Finsler metrics and hence new Finsler solutions for Hilbert's fourth problem. Although, Lots of new projectively flat Finsler metrics can be constructed, we will mention only two examples.

For simplicity we consider the following special case.

Corollary 5.1. *Putting $c_{ij} = \lambda\delta_{ij}$, we have*

$$F = \sqrt{\frac{4\lambda(\lambda|x|^2 + \langle c', x\rangle + c)|y|^2 - 4\lambda^2\langle x, y\rangle^2 - 4\lambda\langle c', y\rangle\langle x, y\rangle - \langle c', y\rangle^2}{4(\lambda|x|^2 + \langle c', x\rangle + c)^2}}, \tag{5.1}$$

is a class of projectively flat metrics with the projective factor

$$\beta = -\frac{2\lambda\langle x, y\rangle + \langle c', y\rangle}{2(\lambda|x|^2 + \langle c', x\rangle + c)}.$$

By making use of the above corollary, since β is closed one form on M and F is projectively flat Riemannian metric, then we have the following example of projectively flat Finsler metric.

Example 1. The class of metrics

$$\overline{F} = \frac{\sqrt{4\lambda(\lambda|x|^2 + \langle c', x\rangle + c)|y|^2 - 4\lambda^2\langle x, y\rangle^2 - 4\lambda\langle c', y\rangle\langle x, y\rangle - \langle c', y\rangle^2} + (2\lambda\langle x, y\rangle + \langle c', y\rangle)}{2(\lambda|x|^2 + \langle c', x\rangle + c)}$$

is new class of projectively flat Finsler metrics. Where

$$\overline{G}^i = P(x, y)y^i, \quad P(x, y) = -\frac{2\lambda\langle x, y\rangle + \langle c', y\rangle}{2(\lambda|x|^2 + \langle c', x\rangle + c)} + \left(F - \frac{\lambda|y|^2}{2F(\lambda|x|^2 + \langle c', x\rangle + c)}\right).$$

Consequently, we have new Finsler solutions for Hilbert's fourth problem.

By the help of [8] (Example 8.2.2, Page 156), we have another Finsler solution for Hilbert's fourth problem as follows.

Example 2. The metric

$$\begin{aligned}\Theta(x, y) &= \frac{\langle c', y\rangle\langle c', x\rangle - 4\lambda\langle x, y\rangle}{4c\lambda|x|^2 - c^2}\\ &+ \frac{\sqrt{16\lambda^2c^2(\langle x, y\rangle^2 - |x|^2|y|^2) + \langle c', y\rangle^2(\langle c', x\rangle^2 + 4\lambda|x|^2 - c^2) - 8\lambda c\langle c', y\rangle\langle c', x\rangle\langle x, y\rangle + 4\lambda c^3|y|^2}}{4c\lambda|x|^2 - c^2}\end{aligned}$$

is Funk metric and, moreover, it is projectively flat with the projective factor $P = \frac{\Theta(x,y)}{2}$. Thus $\Theta(x, y)$ is projectively flat with constant flag curvature $-\frac{1}{4}$.

Proof. Using (5.1), we have

$$F(0, y) = \phi(y) = \sqrt{\frac{4\lambda c|y|^2 - \langle c', y\rangle^2}{4c^2}}.$$

Define

$$\Theta(x,y) = \phi(y+\Theta(x,y)x) = \sqrt{\frac{4\lambda c|y+\Theta(x,y)x|^2 - \langle c', y+\Theta(x,y)x\rangle^2}{4c^2}}.$$

Squaring both sides of the above equation and solving it for Θ, we get the required formula. Since $\phi(y)$ is a Minkowski norm, then $\Theta(x,y)$ is Funk metric and it is projectively flat metric with the projective factor $P = \frac{\Theta(x,y)}{2}$. □

The following example shows a one form deformation of a non flat spray which is not metrizable.

Example 3.
Let $M = \{(x^1,x^2) \in \mathbb{R}^2 : x^2 > 2\}$ and S_0 be a spray given by the coefficients

$$G^1_0 := \frac{(y^1)^2}{2x^2}, \qquad G^2_0 := 0,$$

and take $\beta = y^1 + y^2$. Now consider the deformation $S = S_0 - 2\beta C$. The new coefficients are given by

$$G^1 := \frac{(y^1)^2}{2x^2} + y^1(y^1+y^2), \qquad G^2 := y^2(y^1+y^2),$$

The spray S is isotropic and the coefficients of the nonlinear connection are given by

$$N^1_1 = \frac{y^1}{x^2} + 2y^1 + y^2, \quad N^2_1 = y^1, \quad N^1_2 = y^2, \quad N^2_2 = y^1 + 2y^2.$$

The horizontal basis is $\{h_1, h_2\}$ where

$$\begin{aligned} h_1 &= \frac{\partial}{\partial x^1} - \left(\frac{y^1}{x^2} + 2y^1 + y^2\right)\frac{\partial}{\partial y^1} - y^2\frac{\partial}{\partial y^2}, \\ h_2 &= \frac{\partial}{\partial x^2} - y^1\frac{\partial}{\partial y^1} - (y^1+2y^2)\frac{\partial}{\partial y^2}. \end{aligned}$$

We have

$$\begin{aligned} v_1 := [[h_1,h_2],h_1] &= -\left(\frac{((x^2)^2y^1 + (x^2)^2y2 + x^2y^1 - y^2x^2 + y^1}{(x^2)^2}\right)\frac{\partial}{\partial y^1} \\ &\quad + \left(\frac{(x^2)^2y^1(x^2)^2y^2 + 2x^2y^1 + x^2y^2 + y^1}{(x^2)^2}\right)\frac{\partial}{\partial y^2} \\ v_2 := \big[[h_1,h_2],h_2\big] &= -\left(\frac{(x^2)^3y^1 + (x^2)^3y^2 + 2y^1}{(x^2)^3}\right)\frac{\partial}{\partial y^1} \\ &\quad + \left(\frac{(x^2)^2y^1 + (x^2)^2y^2 - x^2y^1 + 2y^1}{(x^2)^2}\right)\frac{\partial}{\partial y^2}. \end{aligned}$$

Being v_1 and v_2 linearly independent we have $\mathcal{H} = Span\{h_1,h_2,v_1,v_2\} = T\mathcal{T}M$, where $\mathcal{H}$ is the holonomy distribution generated by the horizontal vectors and their successive Lie brackets. Consequently, the Liouville vector field $C \in \mathcal{H}$ hence the spray is not metrizable.

The following example introduces a one form deformation of a flat spray which is not metrizable.

Example 4.
Let $M = \{(x^1,x^2) \in \mathbb{R}^2 : x^2 > 0\}$ and S_0 be a flat spray. So the coefficients are given by

$$G^1_0 = G^2_0 = 0,$$

and take $\beta = y^1 + y^2$. Now consider the deformation $S = S_0 - 2\beta C$. The new coefficients are given by

$$G^1 := y^1(y^1+y^2), \qquad G^2 := y^2(y^1+y^2),$$

The spray S is isotropic and the coefficients of the nonlinear connection are given by

$$N^1_1 = 2y^1 + y^2, \quad N^2_1 = y^1, \quad N^1_2 = y^2, \quad N^2_2 = y^1 + 2y^2.$$

The horizontal basis is $\{h_1, h_2\}$ where

$$h_1 = \frac{\partial}{\partial x^1} - \left(2y^1 + y^2\right)\frac{\partial}{\partial y^1} - y^2\frac{\partial}{\partial y^2},$$
$$h_2 = \frac{\partial}{\partial x^2} - y^1\frac{\partial}{\partial y^1} - (y^1 + 2y^2)\frac{\partial}{\partial y^2}.$$

We have

$$v_1 := [h_1, h_2] = -\left(y^1 + y^2\right)\frac{\partial}{\partial y^1} + \left(y^1 + y^2\right)\frac{\partial}{\partial y^2}.$$

The successive Lie brackets of h_1 and h_2 produce no more linearly independent vectors and hence the holonomy distribution $\mathcal{H} = Span\{h_1, h_2, v_1\}$. The metric freedom [10] of S is unique. Now we can check if we have regular energy function metricizes S or not.

The spray S is Finsler metrizable if there exists a function E satisfying the following system of partial differential equations

$$\mathcal{L}_C E = 2E, \qquad d_h E = 0,$$

which can be written in the form

$$y_1\dot{\partial}_1 E + y_2\dot{\partial}_2 E - 2E = 0,$$
$$\frac{\partial E}{\partial x^1} - \left(2y^1 + y^2\right)\frac{\partial E}{\partial y^1} - y^2\frac{\partial E}{\partial y^2} = 0,$$
$$\frac{\partial E}{\partial x^2} - y^1\frac{\partial E}{\partial y^1} - (y^1 + 2y^2)\frac{\partial E}{\partial y^2} = 0,$$
$$-\left(y^1 + y^2\right)\frac{\partial E}{\partial y^1} + \left(y^1 + y^2\right)\frac{\partial E}{\partial y^2} = 0.$$

The above system has the solution

$$E = C_1 e^{4(x^1 + x^2)}(y^1 + y^2)^2.$$

The matrix (g_{ij}) associated with E is singular and hence the spray is not metrizable. Here in this example β is closed but $det(\partial_i b_j + b_i b_j) = 0$.

Acknowledgement

The author would like to express his deep gratitude to Professor Zoltán Muzsnay (University of Debrecen) for his valuable discussions and suggestions.

References

[1] W. Ambrose, R. S. Palais and I. M. Singer, *Sprays*, Acad. Brasil. Ciencas, **32**, (1960) 163–178.
[2] D. Bao and C. Robles, *Ricci and flag curvatures in Finsler geometry*, in A sampler of Riemann - Finsler geometry, edited by D. Bao et al., Math. Sci. Res. Inst. Publ. 50, Cambridge Univ. Press, Cambridge, 2004, 197–259.
[3] L. Berwald, *Überdien-dimensionalen Geometrien konstanter Krümmung, in denen die Geraden die Kürzesten sind*, Math. Z., **30** (1929), 449–469.
[4] I. Bucataru and Z. Muzsnay, *Projective and Finsler metrizability: parameterization-rigidity of the geodesics*, Int. J. Math. **23**, 9 (2012).
[5] I. Bucataru and Z. Muzsnay, *Projective metrizability and formal integrability*, Symmetry, Integrability and Geometry: Methods and Applications, 7(2011).
[6] I. Bucataru and Z. Muzsnay, *Sprays metrizable by Finsler functions of constant flag curvature*, Diff. Geom. Appl., **31**, 3 (2013), 405-415.
[7] I. Bucataru and Z. Muzsnay, *Metrizable isotropic second-order differential equations and Hilbert's fourth problem*, J. Aust. Math. Soc., **97** (2014), 27–47.
[8] S. S. Chern and Z. Shen, *Riemann-Finsler Geometry*, World Scientific Publishers 2004.
[9] M. Crampin, *On the inverse problem for sprays*, Publ. Math. Debrecen, **70** (2007), 319–335.
[10] S. G. Elgendi and Z. Muzsnay, *Freedom of h (2)-variationality and metrizability of sprays*, Diff. Geom. Appl., **54** A (2017), 194–207.
[11] G. Hamel, *Über die Geometrien in denen die Geraden die Kürzesten sind*, Math. Ann., **57**(1903), 231–264.
[12] D. Krupka and A. E. Sattarov, *The inverse problem of the calculus of variations for Finsler structures*, Math. Slovaca, **35** (1985), 217–222.
[13] O. Krupková, *Variational metric structures* Publ. Math. Debrecen **62** (2003), no. 3-4, 461–495.
[14] B. Li, *On the classification of projectively flat Finsler metrics with constant flag curvature*, Adv. Math., **257** (2014), 266–284.
[15] Z. Shen, *Differential geometry of spray and Finsler spaces*, Springer, 2001.

[16] J. Szilasi and S. Vattamany, *On the Finsler-metrizabilities of spray manifolds*, Period. Math. Hungar., **44** (2002), 81–100.

S.G. Elgendi, Department of Mathematics, Faculty of Science, Benha University, Egypt
E-mail address: salah.ali@fsci.bu.edu.eg