Alternative analysis of a fluid queue driven by Markovian queue

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Abstract

In this paper, we consider a fluid queue with an infinite buffer capacity which is both filled and depleted by fluid at constant rats. These rats are uniquely determined by the number of customers in an *M/M/1/N* queue with constants arrival and service rates. An alternative approach to obtain analytical expression for the joint stationary distribution of the buffer level and the state of *M/M/1/N* queue is given. Through our approach we obtain the determinant of a tridiagonal matrix in terms of the roots of Chebychev's polynomial of second kind. Moreover, we illustrate the effectiveness of the derived formula through graphs and numerical discussion.

KEYWORDS: Fluid queue, *M/M/1/N* queue, stationary state, buffer content distribution, Laplace transform, Chebychev's polynomial.

MATHEMATICS SUBJECT CLASSIFICATION: 60K25, 90B22, 68M20

1. Introduction

In the literature, Stochastic fluid flow models become one of the important topics in queueing theory which have a wide area of applications in many categories as telecommunications, computer systems and manufacturing models, for example see among others Anick et al.(1982) Coffman et al. (1991), Meitra (1988) Stern and Elwalid (1991). In such models, the bursts of data are usually transmitted with high-speed networks in packets or cells. Therefore, the use of fluid models is very useful, since the variations on the cell level are almost negligible compared to those on the more important burst level. In Fluid models it can manipulate models which have continuous customer or stream of nature it prove to be efficient for studying performance analysis of telecommunication and manufacturing models. Many authors and researchers analyze fluid models which have infinite state space of the Markov process that modulates the input rate of fluid in buffer as (Adan and Resing (1996), Sericola and Tuffin (1999) and Virtamo and Norros (1994), but a few of them analyze fluid models which have a finite state space as (Lenin and Parthasarathy (2000), and O'Reilly(1986)). Lenin and Parthasarathy (2000) find a closed-form analytical expression for eigenvalues of the underlying tridiagonal matrix and the distribution of the buffer occupancy by using some identities of tridiagonal determinates for general case *N* takes finite value. In O'Reilly (1986) the explicit expression for the density function of the buffer occupancy in steady state is obtained in special case. Our motivation in this paper is to study a fluid queue driven by an *M/M/1/N* queue (Sharma &Gupta (1983), Tarabia (2000)) with the direct approach to find an analytical explicit expression for the stationary distribution of the buffer occupancy for general case *N* takes finite value. In order to achieve it we convert system of differential equations into system of algebraic equations by using Laplace transform technique then computing the invers of the matrix of this model in terms of roots of Chebychev's polynomial then using the partial fraction method to obtain the demand solution. This technique is more efficient and easier to manipulate this problem. We restrict our analysis to a fluid queue has explicit constant input and service rates.

The rest of the paper is organized as follows: In Section 2, notations and preliminaries are provided. The model analysis is discussed in Section 3. In Section 4, we analyze the solution methodology. The numerical illustration is discussed in Section 5. In section 6, it is include the conclusion and future work.

2. Preliminaries

We consider a fluid queue driven by an *M/M/1/N* queue which can be represented by two dimensional Markov process $\{(X(t), Q(t)), t \geq 0\}$. The first component is acted by a continuous time Markov chain $\{X(t), t \geq 0\}$ with arrival rate λ and service rate μ, where $X(t)$ is a random variable denoting the number of customers in the system at a time t and taking the values $S = \{0, 1, 2, ..., N\}$ and let the generator of the process $\{ X(t) \}$ be denoted by D **,** that is

$$D = \begin{pmatrix} -\lambda & \mu & & & \\ \lambda & -(\lambda+\mu) & \mu & & \\ & \ddots & \ddots & \ddots & \\ & & & & \mu \\ & & & \lambda & -\mu \end{pmatrix}_{(N+1)\times(N+1)} ,$$

where $\lambda > 0$ and $\mu > 0$. The second component is acted by a fluid queue with an infinite buffer which has input rate r_j and service rate q_j such that $r_j > q_j$, $r_j > 0$ to avoid the trivial case where the queue remains always empty and $Q(t)$ denotes the content of the buffer at a time t with $Q(t) \geq 0$ where the content of the buffer cannot decrease whenever the reservoir is empty. That is,

$$\frac{dQ(t)}{dt} = \begin{cases} 0, & if \quad Q(t) = 0 \ and \ r_j - q_j < 0 \\ r_j - q_j \ , & else. \end{cases}$$

Let the drifts of fluid queue represent the difference between the input and service rates

$$d_j = r_j - q_j \quad , \quad j \in S \text{ and } S = \{0,1,2,...,N\},$$

we take in attention the stationary behavior of that fluid queue so, we suppose the following stability condition:

$$\sum_{j=0}^{N} p_j (r_j - q_j) < 0,$$

where p_j be steady state probabilities of the background *M/M/1/N* queue. Also, through our analysis we suppose that this stability condition is satisfied. We define the buffer occupancy distribution $F_j(t,u)$ as

$$F_j(t,u) = prob\{X(t) = j, Q(t) \leq u\}, \ j \in S \ , \ u \geq 0,$$

where $F_j(t,u)$ denotes the probability that the regulating process is in state j and the buffer content does not exceed u at a time t . In steady state case

$$F_j(u) = \lim_{t \to \infty} prob\{X(t) = j, Q(t) \leq u\},$$

with the boundary conditions are $F_0(0) = 1$ and $F_j(0) = 0$.

3. Model analysis

For any fluid queue and for $u > 0$, we have

$$\frac{\partial F_j(t,u)}{\partial t} = \lambda_{j-1} F_{j-1}(t,u) + \mu_{j+1} F_{j+1}(t,u) - (r_j - q_j) \frac{\partial F_j(t,u)}{\partial u} - (\lambda_j + \mu_j) F_j(t,u), \qquad (1)$$

which is called the govern differential equation for all Markovian fluid queues.

Special cases:

(i) If $j = 0$, equation (1) becomes:

$$\frac{\partial F_0(t,u)}{\partial t} = \mu_1 F_1(t,u) - (r_0 - q_0)\frac{\partial F_0(t,u)}{\partial u} - \lambda_0 F_0(t,u) \tag{2}$$

(ii) If $j = N$, equation (1) becomes:

$$\frac{\partial F_N(t,u)}{\partial t} = \lambda_{N-1} F_{N-1}(t,u) + \mu_{N+1} F_{N+1}(t,u) - (r_N - q_N)\frac{\partial F_N(t,u)}{\partial u} - (\lambda_N + \mu_N) F_N(t,u) \tag{3}$$

Consider the following conditions

(i) $r_j - q_j = r - q$ for all $j \in S$.

(ii) $\lim_{u\to\infty} F_j(u) = p_j$.

(iii) The Kolmogorov forward equations for the two dimensional Markov process $\{X(t), Q(t), t \geq 0\}$ in steady state i.e. $\frac{\partial F_j(t,u)}{\partial t} = 0$.

Applying conditions (i) and (iii) on equations (1)-(3), we get

$$\frac{dF_0(u)}{du} = \frac{-\lambda}{(r-q)} F_0(u) + \frac{\mu}{(r-q)} F_1(u) \quad , j = 0 \tag{4}$$

$$\frac{dF_j(u)}{du} = \frac{\lambda}{(r-q)} F_{j-1}(u) - \frac{(\lambda+\mu)}{(r-q)} F_j(u) + \frac{\mu}{(r-q)} F_{j+1}(u) \quad , \quad 1 \leq j \leq N-1 \tag{5}$$

$$\frac{dF_N(u)}{du} = \frac{\lambda}{(r-q)} F_{N-1}(u) - \frac{\mu}{(r-q)} F_N(u) \quad , \quad j = N \tag{6}$$

and $u \geq 0$ for all states $j \in S$. Define

$$\psi_j(\theta) = L[F_j(u)] = \int_0^\infty e^{-\theta u} F_j(u)\,du$$

After taking Laplace transform and using the boundary conditions and making some calculations equations (4)-(6) become:

$$[\theta + \frac{\lambda}{r-q}]\psi_0(\theta) - \frac{\mu}{r-q}\psi_1(\theta) = 1 \quad , \quad j = 0 \tag{7}$$

$$\frac{-\lambda}{r-q}\psi_{j-1}(\theta) + [\theta + \frac{\lambda+\mu}{r-q}]\psi_j(\theta) - \frac{\mu}{r-q}\psi_{j+1}(\theta) = 0 \quad , \quad 1 \leq j \leq N-1 \tag{8}$$

$$\frac{-\lambda}{r-q}\psi_{N-1}(\theta) + [\theta + \frac{\mu}{r-q}]\psi_N(\theta) = 0 \quad , \quad j = N \tag{9}$$

which can be written in the matrix notation as

$$A.\psi(\theta) = B$$

where

$$\psi(\theta) = [\psi_0(\theta), \psi_1(\theta), \ldots, \psi_N(\theta)]^T{}_{(N+1)\times 1} \quad \text{and} \quad B = [1, 0, \ldots, 0]^T{}_{(N+1)\times 1}, \quad \text{hence}$$

$$A = \begin{pmatrix} b-c & c & & & \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & & & c \\ & & & a & b-a \end{pmatrix}_{(N+1)\times(N+1)},$$

where

$$a = \frac{-\lambda}{r-q}, \; b = \theta + \frac{(\lambda+\mu)}{r-q}, \text{ and } \; c = \frac{-\mu}{r-q} \tag{10}$$

4. Solution methodology

To derive the solution for the above system we can write

$$\psi(\theta) = A^{-1}B$$

or

$$\psi(\theta) = \frac{A^*}{\det(A)}, \tag{11}$$

where $A^* = [A_{0j}]^T{}_{(N+1)\times 1}$ $, 0 \le j \le N$ and A_{0j} is the minor element in the 0^{th} column and j^{th} row of the matrix A.

THEOREM 1.

For any nonnegative integer N, the value of the determinate of the matrix A is given as:

$$\det(A) = \theta(\alpha-\beta)^{-1}(\alpha^{N+1} - \beta^{N+1}),$$

where α and β are the eigenvalues of the matrix $\begin{pmatrix} b & ac \\ 1 & 0 \end{pmatrix}$ and are given as:

$$\alpha = \frac{\sqrt{\lambda\mu}}{r-q} e^{J\varphi}, \quad \beta = \frac{\sqrt{\lambda\mu}}{r-q} e^{J\varphi}, \quad J = \sqrt{-1} \text{ and } \varphi = \cos^{-1}\left(\frac{b}{2\sqrt{a.c}}\right)$$

Proof.

We know that the determinate of matrix A has the following formula:

$$\det(A) = (b-c).D_N - (ac)D_{N-1} \tag{12}$$

where

$$D_i = \det\begin{pmatrix} b & c & & & \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & & & c \\ & & & a & b-a \end{pmatrix}_{(i)\times(i)}, \qquad i = 3,4,\ldots N \tag{13}$$

Also

$$\begin{pmatrix} D_m \\ D_{m-1} \end{pmatrix} = S\begin{pmatrix} D_{m-1} \\ D_{m-2} \end{pmatrix}$$

where

$$S = \begin{pmatrix} b & ac \\ 1 & 0 \end{pmatrix}$$

Recursively we have

$$\begin{pmatrix} D_m \\ D_{m-1} \end{pmatrix} = S^m\begin{pmatrix} D_2 \\ D_1 \end{pmatrix}, \qquad m = 2,3,\ldots,N$$

With $D_1 = b - a$ and $D_2 = (b-a)(b-c) - ac$.

Write the matrix S in the spectral form as:

$$S = (\alpha-\beta)^{-1}\begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} -1 & -\beta \\ -1 & \alpha \end{pmatrix},$$

where

$$\alpha,\beta = \frac{b \pm \sqrt{b^2-4ac}}{2}. \tag{14}$$

After some calculation D_i has the following form:

$$D_i = (\alpha-\beta)^{-1}[(\alpha^{i+1}-\beta^{i+1}) - \frac{\lambda}{r-q}(\alpha^i - \beta^i)], \qquad i = 0,1,\ldots,N \ . \tag{15}$$

Hence,

$$D_N = (\alpha-\beta)^{-1}[(\alpha^{N+1}-\beta^{N+1}) - \frac{\lambda}{r-q}(\alpha^N - \beta^N)]$$

Substituting from (15) for $i = N-1, N$ into equation (12) and using the characteristic equation of the matrix S after some calculations, we can complete the proof.

Lemma 1.

For any state j, A_{0j} *has the following form*

$$A_{0j} = (-1)^j a^j D_{N-j} \qquad , \quad j \in S$$

where the D_{N-j} *is given as equation (13) with* $i = N - j$.

Proof.

Clearly, we have $A_{00} = D_N$, $A_{01} = -a.D_{N-1}$, $A_{02} = a^2 D_{N-2}$ and $A_{03} = -a^3 D_{N-3}$

So, in general case we obtain

$$A_{0j} = (-1)^j a^j D_{N-j} \ ,$$

Theorem 2.

For every $j \in S$ *, the buffer occupancy distribution is:*

$$F_j(u) = \frac{(1-\rho)\rho^j}{(1-\rho^{N+1})} - \frac{\rho^{\frac{j+1}{2}}}{(N+1)} \sum_{i=1}^{N} \frac{\sin\upsilon[\sin j\upsilon - \sqrt{\rho}\sin(j+1)\upsilon]}{(1+\rho-2\sqrt{\rho}\cos\upsilon)} e^{[\frac{-\lambda}{r-q} - \frac{\mu}{r-q} + 2\frac{\sqrt{\lambda\mu}}{r-q}\cos\upsilon]u} \tag{16}$$

where $\rho = \dfrac{\lambda}{\mu}$ *is traffic intensity and* $\upsilon = \dfrac{\pi i}{N+1}$

Proof.

Writing

$$(\alpha^{N+1} - \beta^{N+1}) = (\alpha - \beta)\sum_{i=0}^{N} \alpha^k \beta^{N-k} \tag{17}$$

also,

$$\sum_{i=0}^{N} \alpha^k \beta^{N-k} = \prod_{i=1}^{N} (\alpha + \beta - 2y_{N,i}\sqrt{\alpha\beta}) \ , \tag{18}$$

where

$$y_{N,i} = \cos(\upsilon) \ , \ \upsilon = \frac{\pi i}{N+1} \ i = 1, 2, ..., k, ..., N \tag{19}$$

are the N roots of the N^{th} degree Chebychev's polynomial of the second kind.

These roots are known to be real and distinct see (Abromowitz and Stegun(1970))

So, we can rewrite the denominator of $\psi(\theta)$ given in equation (11) as:

$$\det(A) = \theta \prod_{i=1}^{N} (\alpha + \beta - 2y_{N,i}\sqrt{\alpha\beta}) \tag{20}$$

Substituting from Eq. (20) in Eq. (11) and using the partial fraction method, we obtain for any $= 0,1,2, \dots, N$:

$$\psi_j(\theta) = \frac{c_0}{\theta} + \sum_{k=1}^{N} \frac{c_k}{(\alpha + \beta - 2y_{N,k}\sqrt{\alpha\beta})} \tag{21}$$

Such that $c_0, c_1, c_2, \dots, c_N$ are unknown coefficients need to be computed.

To obtain c_0 , multiply equation (20) by θ and take the limit when θ tends to zero so, we get

$$c_0 = \lim_{\theta \to 0} \theta . \psi_j(\theta) \tag{22}$$

Substituting in equation (14) by $\theta \to 0$ and after some calculation, we have

$$\alpha = \frac{\lambda}{r-q} \quad \text{and} \quad \beta = \frac{\mu}{r-q} \tag{23}$$

Substituting from Eq.(23) and using Theorem 1, and Lemma 1 after simple calculations, we have

$$c_0 = \frac{(1-\rho)\rho^j}{(1-\rho^{N+1})} \tag{24}$$

Similarly, we can obtain c_k as the following.

Multiply equation (21) by $[\theta(r-q) + \lambda + \mu - 2y_{N,k}\sqrt{\lambda\mu}]$ and take the limit when θ tends to θ_k

Where

$$\theta_k = \frac{-\lambda - \mu + 2y_{N,k}\sqrt{\lambda\mu}}{r-q} , \qquad y_{N,k} = \cos(\frac{\pi k}{N+1}) \tag{25}$$

So, we have

$$c_k = \frac{1}{r-q} \{ \lim_{\theta \to \theta_k} ([\theta(r-q) + \lambda + \mu - 2y_{N,k}\sqrt{\lambda\mu}]\psi_j(\theta)) \} \tag{26}$$

$$c_k = [\frac{(-1)^j (\alpha)^j}{r-q}][\frac{\lim_{\theta\to\theta_k}[\frac{D_{N-j}}{\alpha-\beta}]}{\lim_{\theta\to\theta_k}[\frac{\theta(\alpha-\beta)\prod_{i=1}^{N}\alpha+\beta-2y_{N,i}\sqrt{\alpha\beta}}{\theta(r-q)+\lambda+\mu-2y_{N,k}\sqrt{\lambda\mu}}]}] \tag{27}$$

$$\lim_{\theta\to\theta_k}[\frac{\theta(\alpha-\beta)\prod_{i=1}^{N}\alpha+\beta-2y_{N,i}\sqrt{\alpha\beta}}{\theta(r-q)+\lambda+\mu-2y_{N,k}\sqrt{\lambda\mu}}] = \frac{1}{(r-q)^N}[(\lim_{\theta\to\theta_k}\theta_k)(\lim_{\theta\to\theta_k}(\alpha-\beta)(\lim_{\theta\to\theta_k}\prod_{\substack{i=1\\ i\neq k}}^{N}\theta(r-q)+\lambda+\mu-2y_{N,i}\sqrt{\lambda\mu})] \tag{28}$$

where

$$\lim_{\theta\to\theta_k}\theta_k = \frac{-\mu(1+\rho-2y_{N,k}\sqrt{\rho})}{r-q} \tag{29}$$

and $$\lim_{\theta\to\theta_k}(\alpha-\beta) = \frac{2J\mu\sqrt{\rho}}{r-q}\sin\upsilon \quad ,\upsilon = \frac{\pi k}{N+1} \quad \text{and } J=\sqrt{-1} \tag{30}$$

Using Chebyshev’s polynomial definition and after some simplification, we get

$$\prod_{\substack{i=1\\ i\neq k}}^{N}(y_{N,k}-y_{N,i}) = 2^{1-N}U'_N(y_{N,k}) = \frac{2^{1-N}(-1)^{k+1}(N+1)}{\sin^2\upsilon} \tag{31}$$

So,

$$\lim_{\theta\to\theta_k}\prod_{\substack{i=1\\ i\neq k}}^{N}\theta(r-q)+\lambda+\mu-2y_{N,i}\sqrt{\lambda\mu} = \frac{(-1)^{k+1}(N+1)(\sqrt{\rho})^{N-1}\mu^{N-1}}{\sin^2\upsilon} \tag{32}$$

Substituting from Eqs. (30)-(32) in (28) , we get

$$\lim_{\theta\to\theta_k}[\frac{\theta(\alpha-\beta)\prod_{i=1}^{N}x+y-2y_{N,i}\sqrt{\alpha\beta}}{\theta(r-q)+\lambda+\mu-2y_{N,k}\sqrt{\lambda\mu}}] = \frac{2J(-1)^k\rho^{\frac{N}{2}}\mu^{N+1}(N+1)(1+\rho-2y_{N,k}\sqrt{\rho})}{(r-q)^{N+2}\sin\upsilon} \tag{33}$$

Similarly,

$$\lim_{\theta\to\theta_k}[\frac{D_{N-j}}{\alpha-\beta}] = \lim_{\theta\to\theta_k}[\alpha^{N-j+1}-\beta^{N-j+1}-\frac{\lambda}{r-q}(\alpha^{N-j}-\beta^{N-j})] \tag{34}$$

where

$$\lim_{\theta\to\theta_k}[\alpha^{N-j+1}-\beta^{N-j+1}]=2J(-1)^{k+1}(\frac{\mu\sqrt{\rho}}{r-q})^{N-j+1}\sin(j)\upsilon \tag{35}$$

Also,

$$\lim_{\theta\to\theta_k}[\alpha^{N-j}-\beta^{N-j}]=2J(-1)^{k+1}(\frac{\mu\sqrt{\rho}}{r-q})^{N-j}\sin(j+1)\upsilon \tag{36}$$

Substituting (39) and (40) in (38) and after some simplification we get,

$$\lim_{\theta\to\theta_k}[\frac{D_{N-j}}{\alpha-\beta}]=\frac{2J(-1)^{k+1}\mu^{N-j+1}\rho^{\frac{N-j+1}{2}}[\sin j\upsilon-\sqrt{\rho}\sin(j+1)\upsilon]}{(r-q)^{N-j+1}} \tag{37}$$

Substituting Eqs. (35)- (36) and (37) in (26) after some simplification, we obtain

$$c_k=\frac{-\rho^{\frac{j+1}{2}}\sin\upsilon[\sin j\upsilon-\sqrt{\rho}\sin(j+1)\upsilon]}{(N+1)(1+\rho-2\sqrt{\rho}\cos\upsilon)} \tag{38}$$

Substituting from Eqs. (24) and (38) and then take inverse of Laplace transform, we can obtain easily:

$$F_j(u)=\frac{(1-\rho)\rho^j}{(1-\rho^{N+1})}-\frac{\rho^{\frac{j+1}{2}}}{(N+1)}\sum_{k=1}^{N}\frac{\sin\upsilon[\sin j\upsilon-\sqrt{\rho}\sin(j+1)\upsilon]}{(1+\rho-2\sqrt{\rho}\cos\upsilon)}e^{[\frac{-\lambda}{r-q}-\frac{\mu}{r-q}+2\frac{\sqrt{\lambda\mu}}{r-q}\cos\upsilon]u} \tag{39}$$

5. NUMIRECAL ILLUSTRATION

In the previous section, we have obtained an explicit expression for the joint stationary distribution of the buffer level and the state of *M/M/1/N* queue. Numerical calculation are made to prove the accuracy and the efficient of our formula. In Figure 1. we plot the relation between $F_j(u)$ and u for different states $j=0,1,2,3,4$ the figure shows that the result carve is according to carve of cumulative function and tends the value $p_j=\frac{(1-\rho)\rho^j}{(1-\rho^{N+1})}$ as $u\to\infty$ it is shown in Figure 1.

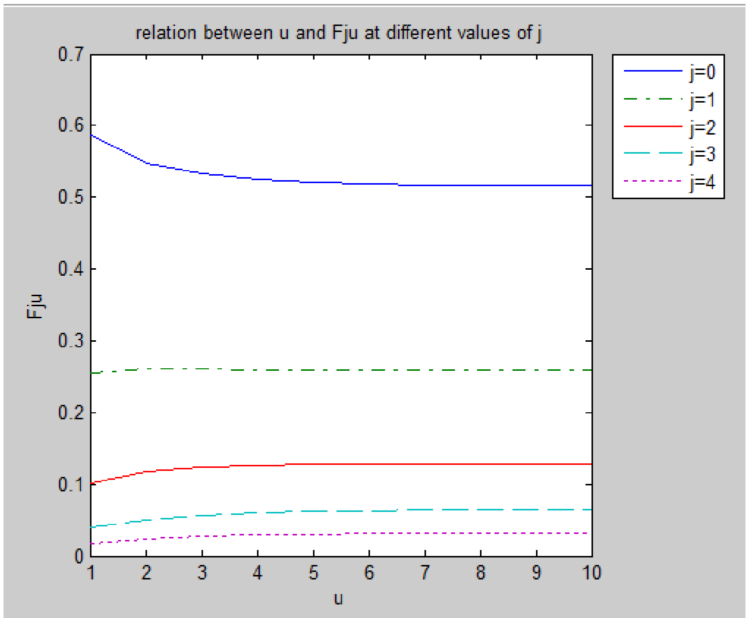

Figure 1: illustrates $F_j(u)$ versus u for different values of j

In Figure 2., we plot the relation between the density function of the buffer content $b(u)$ and u

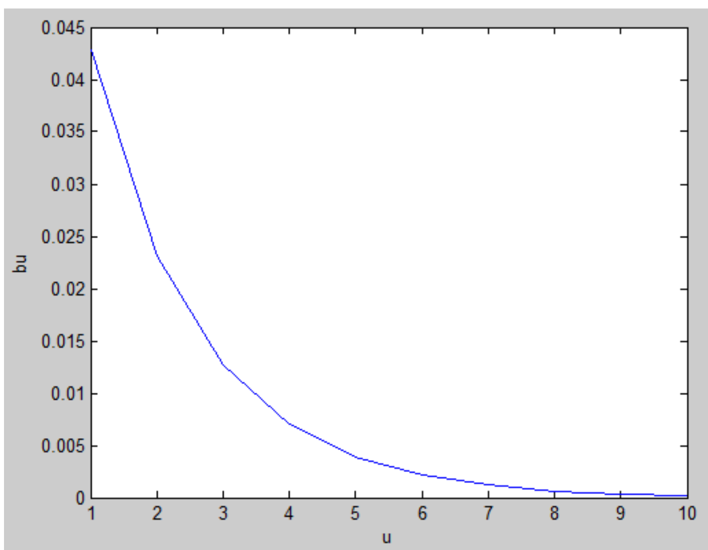

Figure 2: illustrates $b(u)$ versus u

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