Second Order Optimality Conditions for Semilinear Elliptic Optimal Control Problem of Infinite Order with Pointwise Mixed Control-State Constraints

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Abstract

In this paper a semiliner elliptic optimal control problem of infinite order with mixed control-state constraints is studied. The existence of regular Lagrange multipliers, first-order necessary and Second order sufficient conditions are obtained.

Key words: Optimal control, semilinear elliptic equation, infinite order operator, mixed control-state constraints, necessary optimality conditions, second order optimality conditions.

1 Introduction

It is known that in the case of nonlinear equations the first order conditions are not in general sufficient for optimality so that we are going to derive a second order conditions. In this paper, we study an optimal control problem for semilinear elliptic distributed control problem governed by elliptic operator of infinite order with pointwise constraints on the control and the state.

The Caushy Dirchlet problem studied by Dubinski [14, 16]

$$L(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, D^{\gamma} u) = h(x), \qquad x \in \Omega$$

$$D^{|\omega|} u(x)|_{\partial\Omega} = 0, \qquad |\omega| = 0, 1, 2, \cdots$$

The Sobolev space of infinite order which defined by

$$w^{\infty}\{a_{\alpha}, p_{\alpha}\}(\Omega) = \{u(x) \in C_0^{\infty}(\Omega) : p(u) \equiv \sum_{|\alpha|=0}^{\infty} \|D^{\alpha} u\|_{p_{\alpha}}^{p_{\alpha}} < \infty\}$$

where $a_{\alpha} \geq 0$ and $p_{\alpha} \geq 1$ are numerical sequences and $\|.\|_p$ is the canonical norm in the space $L_p(G)$.

Gali et al. [15] presented a set of inequalities defining on optimal control of a system governed by self-adjoint elliptic operators with an infinite number of variables.

Subsequently Lions suggested a problem related to this result but in different direction by taking the case of operators of infinite order with finite dimensions.

Gali has solved this problem, the result has been published in [13].

Moreover, I. M. Gali et. al. [13, 15, 17, 14, 16, 18] presented some control problems generated by both elliptic and hyperbolic linear operator of infinite order with finite number of variables.

El-Zahaby et al [12] obtained the optimal control of problems governed by variational inequalities of infinite order with finite domain.

We refers for instance, to Cases [3] for first-order necessary optimality conditions, Casas, Tröltzsch and Unger [7] for second-order sufficient condition.

The analysis is often simpler for problems with mixed pointwise control-state constraints, since Lagrange multipliers are more regular there. For the elliptic case with quadratic objective and linear equation of infinite order, this obtained by El-Zahaby et. al. [11], and semilinear problem of infinite order with finite dimension, this obtained by El-Zahaby [10].

The existence of regular Lagrange multipliers was discussed by Roch and Troltzsch [20] for elliptic case.

In fact, Meyer and Troltzsch [19] gave sufficient conditions this result was proved by reducing to a problem with pure control constraints.

For the papers which a close connection to our work, we refer to [1, 2, 5, 4, 6] and reference given there in.

2 Some Function Spaces

The embedding problems for non-trivial Sobolev spaces of infinite order are investigated in [8, 9].

An embedding criterian established in terms of the characteristic functions of these space.

In this case

$$W^{\infty}\{a_\alpha, 2\} \subseteq L_2(R^n) \subseteq W^{-\infty}\{a_\alpha, 2\}$$

where,

$$W^{\infty}\{a_\alpha, 2\} = \{\phi \in C^{\infty}(R^n) : \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha \phi\|_2^2 < \infty\}$$

be Sobolev space of infinite order of periodic function defined on all of R^n and $W^{-\infty}\{a_\alpha, 2\}$ denotes their topological dual with respect to $L_2(R^n)$, we recall that $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index for differentiation and $|\alpha| = \alpha_1 + \cdots + \alpha_n$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots (\partial x_n^{\alpha_n})}, \qquad a_\alpha > 0,$$

is a numerical sequence, and $\|.\|_2$ is the canonical norm in the space $L_2(R^n)$, (all functions are assumed to be real value).

3 Our Results

In the study of semilinear elliptic control problem of infinite order and pointwise state and control constraints

$$(P) \quad \begin{cases} \min J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{k}{2}\|u\|^2_{L^2(\Omega)}, & (3.1) \\ \text{subject to} & \\ Ay + d(x,y) = u \qquad \text{in} \quad \Omega & (3.2) \\ y^{|w|}|_\Gamma = 0 \qquad |w| = 0, 1, 2, \cdots & \\ \text{and the mixed control-state constraints} & \\ y_a(x) \le \lambda u(x) + y(x) \le y_b(x) \qquad \text{a.e in} \quad \Omega & (3.3) \end{cases}$$

where A is the elliptic operator of infinite order $A \in L(W^{\infty}\{a_\alpha, 2\}, W^{-\infty}\{a_\alpha, 2\})$ for which has self-adjoint closure

$$Ay(x) = \sum_{|\alpha|=0}^{0} (-1)^{|\alpha|} a_\alpha D^{2\alpha} y, \qquad a_\alpha > 0 \tag{3.4}$$

we introduce the following continuous bilinear form

$$a(u,v) = \sum_{|\alpha|=0}^{0} \Big((-1)^{|\alpha|} a_\alpha D^{2\alpha} u(x), v(x)\Big)_{L^2(R^n)} \qquad \text{on} \qquad W^{\infty}\{a_\alpha, 2\}$$

It is well known that the ellipticity of a is sufficient for the coerciveness of $a(u,v)$ i.e.,

$$a(y,y) \ge \nu \|u\|^2_{W_0^{\infty}\{a_\alpha, 2\}}$$

We adapt the following assumption

Assumption 1:

- The function $y_d \in L^2(\Omega)$ and $k > 0, N \neq 0, \lambda \neq 0$ are real numbers and the bound y_a and y_b are fixed function in $L^\infty(\Omega)$ with $y_a(x) \leq y_b(x)$.
- $\Omega \subset R^N$ be bounded Lipschitz domain with boundary Γ.
- A denotes elliptic operator of infinite order with finite dimension take form (3.4).
- This operator is bounded self-adjoint mapping $W_0^\infty\{a_\alpha, 2\}$ onto $W_0^\infty\{a_\alpha, 2\}$ and satisfy the condition of ellipticity [14]

$$a(y(x), y(x)) = \sum_{|\alpha|=0}^{\infty} (a_\alpha D^\alpha y(x), D^\alpha y(x))_{L^2(R^n)} \geq \nu \|y\|^2_{W^\alpha\{a_\alpha,2\}}, \quad 1 \geq \nu > 0.$$

- The function $d = d(x, y) : \Omega \times R \to R$ is measurable with respect to $x \in \Omega$ for all fixed $y \in R$, and twice continuously differentiable with respect to y, for almost all $x \in \Omega$.
- Moreover, for d is satisfy the boundedness condition of order two with respect to x, there exists $C > 0$ such that
$$\|d(.,0)\|_\infty + \|d_y(.,0)\|_\infty + \|d_{yy}(.,0)\|_\infty \leq C$$
- Further, for $a.a.x \in \Omega$ and $y \in R$, it holds that $d_y(x, y) \geq 0$.
- Also, the derivative of d w.r.t. y up to order two are local Lipschitz condition, i.e. for all $M > 0$ there exists $L_M > 0$ such that d satisfies
$$\|d_{yy}(., y_1) - d_{yy}(., y_2)\|_\infty \leq L_M |y_1 - y_2|$$
for all $y_i \in R$ with $|y_i| \leq M$, $i = 1, 2$.
- There is a subset $E_\Omega \subset \Omega$ of positive measure with $d_y(x, y) > 0$ in $E_\Omega \times R$.

Theorem 3.1. *Under Assumption 1 the semilinear elliptic control problem* (3.2) *admits for every* $u \in L^2(\Omega)$ *a unique solution* $y \in W^\infty\{a_\alpha, 2\} \cap C(\bar{\Omega})$ *with*

$$\|y\|_{W^\infty\{a_\alpha,2\}} + \|y\|_{C(\bar{\Omega})} \leq \|u\|_{L^2(\Omega)}.$$

For the proof we refer to [10].

4 First-Order Conditions

We introduce the control-to-state operator

$$G : L^2(\Omega) \to W^\infty\{a_\alpha, 2\}(\Omega) \cap C(\bar{\Omega}), \qquad u \mapsto y.$$

Let us reformulate the problem (P) with the help of solution operator G to obtain the reduced formulation

$$J(y, u) = J(G(u), u) := \frac{1}{2} \int_\Omega (G(u) - y_d(x))^2 + \frac{k}{2} \int_\Omega u^2(x) dx = f(u)$$

$$y_a(x) \leq \lambda u(x) + G(u)(x) \leq y_b(x) \qquad a.e. \quad in\ \Omega$$

The next theorem states the existence of an optimal solution for (P).

Definition 4.1. A control $\bar{u} \in U_{ad}$ satisfy $y_a(x) \leq \lambda \bar{u}(x) + G(\bar{u})(x) \leq y_b(x)$ in Ω is called a local solution of problem (P) if there exists a $\rho > 0$ such that $f(\bar{u}) \leq f(u)$ for all $u \in U_{ad}$ and $\|u - \bar{u}\| \leq \rho$.

Theorem 4.1. *Let the Assumption 1 be satisfied. If the admissible set is not empty, then Problem* (P) *admits at least one local solution in the sense of Definition 4.1.*

We finish this section by recalling some results about the differentiability of the functionals involved in the control problem. For the detailed proofs the reader is referred to cases and Mateos [4].

Lemma 4.2. *Let Assumption 1 be fulfilled. Then the mapping* $G : L^2(\Omega) \to W^\infty\{a_\alpha, 2\} \cap C(\bar{\Omega})$ *is twice continuously Frechet differentiable. Moreover, for all* $u \in L^2(\Omega)$, $w = G'(u)h$ *is defined as the solution of*

$$\begin{aligned} Aw + d_y(x,y)w = h & \qquad in \quad \Omega \\ w^{|\alpha|}|_\Gamma = 0 & \qquad |\alpha| = 0,1,2,\cdots \end{aligned} \tag{4.1}$$

Furthermore, for every $z \in L^2(\Omega)$, $z = G''(u)[u_1, u_2]$ *is the solution of*

$$\begin{aligned} Az + d_y(x,y)z = -d_{yy}(x,y)y_1y_2 & \qquad in \quad \Omega \\ z^{|w|}|_\Gamma = 0 & \qquad |w| = 0,1,2,\cdots \end{aligned} \tag{4.2}$$

where $y_i = G'(u)u_i$.

The proof can be obtained by using the implicit function theorem.

Since J is twice continuously Frechet differentiable and with differentiability of G, this yields the following Lemma.

Lemma 4.3. *Under Assumption 1,* f *is twice continuously Frechet differentiable from* $L^2(\Omega)$ *to* R*. Its first derivative is given by*

$$f'(u)h = \int_\Omega (ku + p(x))h(x)dx \tag{4.3}$$

where p *solves the adjoint equation*

$$\begin{aligned} Ap + d_y(x,y)p = y - y_d & \qquad in \quad \Omega \\ p^{|w|}|_\Gamma = 0 & \qquad |\alpha| = 0,1,2,\cdots \end{aligned} \tag{4.4}$$

where p *is the adjoint solution and* A *is adjoint operator which take the same form in* (3.4).

For the second derivative, we obtain

$$f''(u)[u_1, u_2] = (y_1, y_2)_{L^2(\Omega)} + k(u_1, u_2) - \int_\Omega d_{yy}(x,y)y_1y_2pdx \tag{4.5}$$

Proof. From definition of the reduced cost functional

$$f(u) := J(G(u), u) = \frac{1}{2}\int_\Omega (G(u) - y_d(x))^2 + \frac{k}{2}\int_\Omega u^2(x)dx$$

We get

$$f'(u)h = (y - y_d, w)_{L^2(\Omega)} + k(u,h)_{L^2(\Omega)},$$

where $y = G(u)$ and $w = G'(u)h$ denotes the weak solution of the linearized equation (4.1) with the right hand side h.

Now, choosing p as a test function in the weak formulation of (4.1) and inserting y in the weak formulation of equation (4.4), we obtain

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} \int_\Omega a_\alpha D^{2\alpha} wpdx + \int_\Omega dy(x,y)wpdx = \int_\Omega hpdx$$

$$\sum_{|\alpha|=0}^{\infty} \int_\Omega a_\alpha (D^\alpha w)(x)(D^\alpha p)(x)dx + \int_\Omega d_y(x,y)wpdx = \int_\Omega hpdx$$

and

$$\sum_{|\alpha|=0}^{\infty}\int_{\Omega}a_{\alpha}(D^{\alpha}p)(x)(D^{\alpha}w)(x)dx+\int_{\Omega}d_{y}(x,y)pwdx=\int_{\Omega}(y-y_{d})wdx$$

Substracting one equation from the other finally yields

$$(y-y_{d},w)_{L^{2}(\Omega)}=(h,p)_{L^{2}(\Omega)}$$

As a simple conclusion, the following expression for the directional derivative of the reduced functional f at $\bar{u}$ in the direction $h\in L^{2}(\Omega)$ yields

$$f'(\bar{u})h=\int_{\Omega}(p(x)+k\bar{u}(x))h(x)dx$$

We obtain the desired necessary optimality condition.

Applying again the Chain rule, we can calculate the second derivative as follow. First, we obtain

$$f'(u)u_{1}=D_{y}J(G(u),u)G'(u)u_{1}+D_{u}J(G(u),u)u_{1}$$

Next, we calculate the direction derivative of $f'(u)u_1$ indirection u_2, we find that

$$f''(u)[u_{1},u_{2}]=J''(y,u)[(y_{1},u_{1}),(y_{2},u_{2})]+D_{y}J(y,u)G''(u)[u_{1},u_{2}].$$

A similar discussion as above, where the abbreviations $z:=G''(u)[u_1,u_2]$ denotes the weak solution of (4.2) with this, we obtain the expression

$$D_{y}J(y,u)z=\int_{\Omega}(y-y_{d})z(x)dx$$

which can be transformed by using the adjoint state p, which is the weak solution to (4.4) hence, we have $(y-y_d,z)_{L^2(\Omega)}=-(d_{yy}(x,y)y_1y_2,p)_{L^2(\Omega)}$. □

Now, we can directly apply the results of C. Meyer and F. Tröltzsch [19] by reducing the problem (P) into a purely control-constrained problem by substitute $\lambda u+G(u)=v$ and defined the associated nonlinear equation

$$\lambda u+G(u)=v \tag{4.6}$$

for a given v is a neighborhood of $\bar{v}=\lambda\bar{u}+G(\bar{u})$.

To prove the existence of solution to (4.3) admits a unique solution in a neighborhood of the optimal solution $\bar{u}$ for all given $v\in L^2(\Omega)$ in a neighborhood of $\bar{v}$.

We need the following a suitable regularity Assumption and applying the implicit function theorem.

To this aim, we define an operator

$$T:L^{2}(\Omega)\times L^{2}(\Omega)\rightarrow L^{2}(\Omega):(u,v)\mapsto v,T(u,v)=\lambda u+G(u)-v.$$

And define a mapping $K:v\rightarrow u$ by $K(v)=u$ that is implicity defined by $T(K(v),v)=0$.

To apply the implicit function theorem, we need that $\frac{\partial T}{\partial u}(\bar{u},\bar{v})u=\lambda u+G'(\bar{u})u$ is invertible.

As in any optimization problem submitted to some nonlinear constraint, we need a regularity assumption to get the first and second order necessary optimality conditions.

Regularity assumption (R)

Let $\lambda\neq 0$ is not an eigenvalue of **-G**, i.e the equation

$$\lambda u+G'(\bar{u})u=0$$

admits only the trivial solution.

We define **G** by $G'(\bar{u})$ with range in $L^2(\Omega)$. By the embedding of the space $W^\infty\{a_\alpha, 2\}$ in $L^2(\Omega)$. **G** is compact and hence **G** is a Fredholm operator that has only countably many eigenvalues accumulation y point at 0. $I : L^2(\Omega) \to L^2(\Omega)$ denotes the identity.

By applying the Fredholm theory we find that the equation

$$\lambda u + G'(\bar{u})u = f, \qquad \lambda \neq 0$$

has a unique solution for each $f \in L^2(\Omega)$ if and only if the equation

$$\lambda u + G'(\bar{u})u = f, \qquad \lambda \neq 0$$

has only trivial solution.

So that regularity assumption (R) holds. Thus, $\dfrac{\partial T}{\partial u}(\bar{u}, \bar{v})$ is continuously invertible by Banach theorem, and hence the implicit function theorem gives the existence of open ball $B_{r_1}(\bar{u}), B_{\rho_1}(\bar{v})$ in $L^2(\Omega)$ s.t. for all $v \in B_{\rho_1}(\bar{v})$, there is exactly one solution $u \in B_{r_1}(\bar{u})$ with $T(u, v) = 0$ and K is twice continuously differentiable in $L^2(\Omega)$ with respect to u.

Lemma 4.4. *Let $K : L^2(\Omega) \to L^2(\Omega)$, $v \mapsto u$ then it's First and second-order derivative are given by*

$$K'(v) = (\lambda I + G'(K(v)))^{-1} \tag{4.7}$$

$$K''(v)[v_1, v_2] = -(\lambda I + G'(K(v)))^{-1} G''(K(v))[K'(v)v_1, K'(v)v_2]. \tag{4.8}$$

Proof. As K is implicitly defined by $T(K(v), v) = 0$, the equation $\lambda K(v) + G(K(v)) = v$ holds true for all v in a neighborhood of $\bar{v}$.

By differentiation to both sides of equation $\lambda K(v) + G(K(v)) = v$ we get

$$\lambda K'(v) + G'(K(v))K'(v) = I \tag{4.9}$$

which implies (4.7). Next, we apply both sides of (4.9) to v_1, and differentiate in direction v_2, we obtain

$$\lambda K''(v)[v_1, v_2] + G'(K(v))K''(v)[v_1, v_2] + G''(K(v))[K'(v)v_1, K'(v)v_2] = 0$$

which gives (4.8). □

By using the control to state operator G and a mapping $K : v \to u$, we can transform the objective functional in the form

$$J(y, u) = J(G(u), u) = f(u) = f(K(v)) =: F(v)$$

where F is defined on $B_\rho(\bar{v})$.

From local optimality of $\bar{u}$ there exist open ball $B_{r_2}(\bar{u})$ in $L^2(\Omega)$ s.t. $f(\bar{u}) < f(u)$ for all $u \in B_{r_2}(\bar{u})$ with $y_a(x) \leq \lambda u(x) + y(x) \leq y_b(x)$ and we can get

$$F(\bar{v}) \leq F(v), \quad \text{for all} \quad v \in L^2(\Omega) \tag{4.10}$$

satisfying

$$y_a(x) \leq v(x) \leq y_b(x) \quad \text{a.e in} \quad \Omega$$

and $\|v - \bar{v}\|_{L^2(\Omega)} < \rho_2$ with a sufficiently small $\rho_2 > 0$, $\rho_2 \leq \rho_1$ and $u = K(v) \in B_{r_2}(\bar{u})$. Thus, we can get a new form of optimization problem in the variable v

$$(PV) \quad \begin{cases} \text{minimize} \quad F(v), \\ \text{subject to} \qquad v \in V_{ad}, v \in B_{\rho_2}(\bar{v}) \end{cases}$$

with an admissible set defined by

$$V_{ad} = \{v \in L^2(\Omega) : y_a(x) \leq v(x) \leq y_b(x) \qquad a.e \qquad in \quad \Omega\}$$

Under Regular assumption, let $\bar{v}$ be a local optimal control for (PV).
Then we have the variational inequality

$$F'(v)(v-\bar{v}) \geq 0 \qquad for\ \ all \qquad v \in V_{ad} \tag{4.11}$$

By the Riesz theorem, the functional $F'(\bar{v}) \in L^2(\Omega)^*$ can be identified with a function from $L^2(\Omega)$ and denote this function by μ. i.e.

$$F'(v)v = \int_\Omega \mu(x)v(x)dx \tag{4.12}$$

and we define nonnegative functions $\mu_a, \mu_b \in L^2(\Omega)$ by

$$\begin{aligned} \mu_a(x) = \mu(x)_+ &= \frac{1}{2}(\mu(x) + |\mu(x)|) \\ \mu_b(x) = \mu(x)_- &= \frac{1}{2}(-\mu(x) + |\mu(x)|) \end{aligned} \tag{4.13}$$

Then, $\mu(x) = \mu_a(x) - \mu_b(x)$ and identifying $F'(\bar{v})$ with μ implies

$$F'(\bar{v}) + \mu_b(x) - \mu_a(x) = 0 \tag{4.14}$$

5 Lagrange Multiplier Rule

We define Lagrangian $\mathcal{L}$ by adding the inequality constraints in the following way

$$\mathcal{L}(y,u,q,\mu_a,\mu_b) := J(y,u) - (Ay + d(x,y) - u, q)_{L^2(R^n)} + (y_a - v, \mu_a)_{L^2(\Omega)} + (v - y_b, \mu_b)_{L^2(\Omega)} \tag{5.1}$$

where μ_a, μ_b are Lagrange multipliers for the control-state constraint, and which defined by (4.13)[21].
Then the equation

$$k\bar{u} + q + \lambda(\mu_b - \mu_a) = 0 \tag{5.2}$$

Can be expressed in the form $\dfrac{\partial \mathcal{L}}{\partial u}(\bar{y},\bar{u},q,\mu_a,\mu_b) = 0$ for all $u \in L^2(\Omega)$.
Moreover, the adjoint equation

$$\begin{aligned} Aq + d_y(x,\bar{y})q &= \bar{y} - y_d + \mu_b - \mu_a \qquad && in \quad \Omega \\ q^{|w|}|_\Gamma &= 0 && |w| = 0,1,2,\cdots \end{aligned} \tag{5.3}$$

is equivalent to the equation

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{y},\bar{u},q,\mu_a,\mu_b) = 0$$

for all $y \in W^\infty\{a_\alpha, 2\}$.
The optimality system can therefore by rewritten in the following form

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial y} = 0, \qquad && \frac{\partial \mathcal{L}}{\partial u} = 0 \\ &\mu_a \geq 0, && \mu_b \geq 0 \\ &(y_a - \bar{v}, \mu_a)_{L^2(\Omega)} = (\bar{v} - y_b, \mu_b)_{L^2(\Omega)} = 0 \end{aligned}$$

where the last two conditions are called complementary slackness conditions.
In the next theorem show that (5.2), (5.3) and complementary slackness conditions follow from the variational inequality (4.11).

Theorem 5.1. *Suppose that $\bar{u}$ is a local solution of optimal control problem, then there exist nonnegative Lagrange multipliers $\mu_a \in L^\infty(\Omega)$ and $\mu_b \in L^\infty(\Omega)$ and adjoint state $p \in H'(\Omega) \cap C(\bar{\Omega})$ such that the condition* (5.2)*, adjoint equation* (5.3) *and complementing slackness conditions are satisfied.*

Proof. (i) adjoint equation and condition (5.2). From $F(v) = f(K(v))$ and by Chain rule we can get

$$F'(\bar{v})v = f'(K(\bar{v}))K'(\bar{v})v$$

and by using (4.14) we find that

$$f'(K(\bar{v}))K'(\bar{v})v + (\mu_b - \mu_a, v)_{L^2(\Omega)} = 0 \qquad \forall \quad v \in L^2(\Omega)$$

By Substituting $u = K'(\bar{v})v$ and $\bar{u} = K(\bar{v})$, one obtains

$$f'(\bar{u})u + (\mu_b - \mu_a, K'(\bar{v})^{-1}u)_{L^2(\Omega)} = 0$$

We introduce equation (4.7) and arrive at

$$f'(\bar{u})u + (\mu_b - \mu_a, (\lambda I + G'(\bar{u}))u)_{L^2(\Omega)} = 0 \tag{5.4}$$

from equation (4.3), shows that the first derivative of f is

$$f'(\bar{u})u = (k\bar{u} + p_1, u)_{L^2(\Omega)}, \tag{5.5}$$

where $p = p_1$ is the solution of (4.4) with $y = \bar{y}$ in the right hand side, we have $p_1 \in W^\infty\{a_\alpha, 2\} \cap C(\bar{\Omega})$ because of $\bar{y} \in W^\infty\{a_\alpha, 2\} \cap C(\bar{\Omega}) \subset L^2(\bar{\Omega})$. For the second term in (5.4), we find

$$(\mu_b - \mu_a, (\lambda I + G'(\bar{u}))u)_{L^2(\Omega)} = \lambda(\mu_b - \mu_a, u)_{L^2(\Omega)} + (\mu_b - \mu_a, w)_{L^2(\Omega)} \tag{5.6}$$

with $w = G'(\bar{u})u$, i.e. w is a solution to linearized equation (4.1) with $y := \bar{y}$ and $h := u$. And we find that

$$(\mu_b - \mu_a, w)_{L^2(\Omega)} = (p_2, u)_{L^2(\Omega)} \tag{5.7}$$

where p_2 solves the adjoint equation

$$\begin{aligned} Ap_2 + d_y(x, \bar{y})p_2 &= \mu_b - \mu_a \qquad && in \quad \Omega \\ p_2^{|w|}|_\Gamma &= 0 && |w| = 0, 1, 2, \cdots \end{aligned} \tag{5.8}$$

From $\mu_b - \mu_a \in L^2(\Omega)$, we have $p_2 \in W^\infty\{a_\alpha, 2\} \cap C(\bar{\Omega})$. Inserting (5.7), (5.6) and (5.5) in (5.4) yields

$$(k\bar{u} + p_1 + p_2 + \lambda(\mu_b - \mu_a), u)_{L^2(\Omega)} = 0 \tag{5.9}$$

where $q = p_1 + p_2$ solves the adjoint equation (5.3). Hence (5.9) is equivalent with (5.2).

(ii) Complementary slackness conditions:
From the variational inequality (4.11) and (4.12) we have

$$F'(\bar{v})(v - \bar{v}) = \int_\Omega (\mu_a - \mu_b)(v - \bar{v})dx \geq o \qquad \forall \quad v \in V_{ad}$$

and thus

$$(\mu_a - \mu_b, \bar{v})_{L^2(\Omega)} = \min(\mu_a - \mu_b, v)_{L^2(\Omega)} = (\mu_a, y_a)_{L^2(\Omega)} - (\mu_b, y_b)_{L^2(\Omega)},$$

since $\mu_a(x) \cdot \mu_b(x) = 0$ and $\mu_a(x), \mu_b(x) \geq 0$ by definition (4.13).
Therefore, if $\mu_a(x) > 0$, we have $\bar{v}(x) = y_a(x)$, while $\mu_b(x) > 0$, we have $\bar{v}(x) = y_b(x)$. This yields

$$(\mu_a, y_a - \bar{v})_{L^2(\Omega)} + (\mu_b, \bar{v} - y_b)_{L^2(\Omega)} = 0 \tag{5.10}$$

Because $\mu_a(x), \mu_b(x) \geq 0$ and $\bar{v} \in V_{ad}$, both addends on the right side of (5.10) are nonpositive and we have

$$(\mu_a, y_a - \bar{v})_{L^2(\Omega)} + (\mu_b, \bar{v} - y_b)_{L^2(\Omega)} = 0$$

This implies the complementary slackness conditions.

□

6 Second-order Sufficient Conditions

We discuss a sufficient second-order optimality condition (SSC) to (PV). For this purpose, we define strongly active set as follows:

Definition 6.1. Let $\tau > 0$ be fixed. Then the strongly active set A_τ is defined as

$$A_\tau(\bar{u}) = \{x \in \Omega : |\mu_a(x) + \mu_b(x)| \geq \tau\}$$

Moreover, the corresponding τ-critical cone with respect to v is defined by

$$\widehat{C}_\tau := \left\{ v \in L^2(\Omega) \;\middle|\; \begin{array}{lll} v(x) = 0 & a.e. \quad in\ A_\tau \\ v(x) \geq 0 & where \quad \bar{v}(x) = y_a(x)\ and x \notin A_\tau \\ v(x) \leq 0 & where \quad \bar{v}(x) = y_b(x)\ and x \notin A_\tau \end{array} \right. \tag{6.1}$$

with $\bar{v} = \lambda\bar{u} + \bar{y}$

The sufficient second-order optimality conditions are given by the equation (6.2) in the next theorem covering the local optimality of $\bar{v}$ [13].

Theorem 6.1. *Let $\bar{v}$ be a feasible control for problem (PV) and satisfies the variational inequality* (4.11)*. Assume that the coercivity condition*

$$F''(\bar{v})v^2 \geq \hat{\delta}\|v\|_{L^2(\Omega)} \qquad \forall\ v \in \widehat{C}_\tau \tag{6.2}$$

is satisfied with $\hat{\delta} > 0$. Then there exist $\hat{\epsilon} > 0$ and $\hat{\sigma} > 0$ s.t. we have the quadratic growth condition holds:

$$F(\bar{v}) + \hat{\sigma}\|v - \bar{v}\|^2_{L^2(\Omega)} \leq F(v)\ if\ \|v - \bar{v}\|_{L^2(\Omega)} < \epsilon\ \forall v \in V_{ad} \tag{6.3}$$

We transfer second-order sufficient optimality condition to the original term y and u. So we mention the following Lemma on $F''(\bar{v})$.

Lemma 6.2. *Let* (R) *is satisfied. Then F is twice continuously Frechet differentiable at $\bar{v}$ and its second derivative is given by*

$$F''(\bar{v})v^2 = \mathcal{L}''_{y,u}(\bar{y}, \bar{u}, q, \mu)(y, u)^2. \tag{6.4}$$

Proof. From Regularity assumption (R) and Since $F(v) = f(K(v))$ we find that F is twice continuously Frechet differentiable in a neighborhood of $\bar{v}$. The chain rule implies

$$F''(v)[v_1, v_2] = f'(K(v))K''(v)[v_1, v_2] + f''(K(v))[K'(v)v_1, K'(v)v_2] \tag{6.5}$$

put $v = \bar{v}$ and thus $K(\bar{v}) = \bar{u}$. We set $v_1 = v_2 = v$ and $K'(v)v_1 = K'(v)v_2 = u$. Hence (6.5) is equivalent to

$$F''(\bar{v})v^2 = f'(\bar{u})K''(\bar{v})v^2 + f''(\bar{u})u^2.$$

In view of (5.4), we have

$$f'(\bar{u})K''(\bar{v})v^2 = -(\mu_b - \mu_a, (\lambda I + G'(\bar{u}))K''(\bar{v})v^2)_{L^2(\Omega)}.$$

From equation (4.8) we have

$$\begin{aligned} F''(\bar{v})v^2 &= (\mu_b - \mu_a, G''(\bar{u})[K'(\bar{v})v, K'(\bar{v})v])_{L^2(\Omega)} + f''(\bar{u})u^2 \\ &= (\mu_b - \mu_a, G''(\bar{u})u^2)_{L^2(\Omega)} + f''(\bar{u})u^2. \end{aligned} \tag{6.6}$$

Since $z = G''(\bar{u})u^2$ solves the equation (4.2) and we can get

$$(\mu_b - \mu_a, z)_{L^2(\Omega)} = -(d_{yy}(x, \bar{y})y^2, p_2)_{L^2(\Omega)}$$

where p_2 is the solution of (5.8) and $y = G'(\bar{u})u$ is the solution of linearized equation (4.1). Thus, together with (4.5) for second derivative of f, (6.6) is transformed into

$$F''(\bar{v})v^2 = \|y\|^2_{L^2(\Omega)} + k\|u\|^2_{L^2(\Omega)} - \int_\Omega d_{yy}(x,\bar{y})y^2(p_1+p_2)dx$$

where p_1 is the solution of (4.4), we have $q = p_1 + p_2$

$$\begin{aligned} F''(\bar{v})v^2 &= \|y\|^2_{L^2(\Omega)} + k\|u\|^2_{L^2(\Omega)} - \int_\Omega d_{yy}(x,\bar{y})y^2 q dx \\ &= J''_{(y,u)}(\bar{y},\bar{u})(y,u)^2 - \int_\Omega d_{yy}(x,\bar{y})y^2 q dx \\ &= \mathcal{L}''_{y,u}(\bar{y},\bar{u},q,\mu)(y,u)^2. \end{aligned}$$

according to the definition of $\mathcal{L}$ in (5.1). □

By using (6.1) we can define the τ-critical cone for the original problem (P) as follows

Definition 6.2. The critical cone associated to (P) is given by

$$C_\tau := \{(y,u) \in W^\infty\{a_\alpha, 2\} \cap C(\bar{\Omega}) \times L^2(\Omega) | y = G'(\bar{u})u \text{ and } \lambda u + y \in \widehat{C}_\tau\}.$$

The second-order sufficient optimality conditions for (P) postulates the existence of some $\delta > 0$ such that

$$\mathcal{L}''_{y,u}(\bar{y},\bar{u},q,\mu)(y,u)^2 \geq \delta\|u\|^2_{L^2(\Omega)} \quad \text{for all } (y,u) \in C_\tau$$

If the pair $(\bar{y},\bar{u})$ satisfies both the first-order necessary conditions and the second-order sufficient condition (SSC) for (P), then $\bar{u}$ is local optimal control as the following results shows.

Theorem 6.3. *Let $(\bar{y},\bar{u})$ satisfy the first-order necessary optimality conditions for (P) and the second-order sufficient conditions, then there exist constant $\epsilon > 0$ and $\sigma > 0$ such that we have the quadratic growth condition*

$$J(y,u) \geq J(\bar{y},\bar{u}) + \sigma\|u-\bar{u}\|^2_{L^2(\Omega)} \quad \forall\ (y,u) \in W^\infty\{a_\alpha,2\} \cap C(\bar{\Omega}) \times L^2(\Omega) \text{ with } \|u-\bar{u}\|^2_{L^2(\Omega)} \leq \epsilon \tag{6.7}$$

where $y = G(u), y_a(x) \leq \lambda u(x) + y(x) \leq y_b(x)$. In particular, $\bar{u}$ is a local optimal control.

Proof. We choose an arbitrary pair $(\eta, h) \in C_\tau$ and define $v := \lambda h + \eta$, $\eta = G'(\bar{u})h$ according to definition of C_τ, we have

$$F''(\bar{v})v^2 = \mathcal{L}''_{y,u}(\bar{y},\bar{u},q,\mu)(\eta,h)^2 \geq \delta\|h\|^2_{L^2(\Omega)} \tag{6.8}$$

where we used (SSC) condition for the last estimate. Due to $h = (\lambda I + G'(\bar{u}))^{-1}v$, (6.8) is equivalent to

$$\begin{aligned} F''(\bar{v})v^2 &\geq \delta\|(\lambda I + G'(\bar{u}))^{-1}v\|^2_{L^2(\Omega)} \geq \delta\left(\frac{1}{\|\lambda I + G'(\bar{u})\|_{\mathcal{L}(L^2(\Omega))}}\|v\|_{L^2(\Omega)}\right)^2 \\ &\geq \delta\|(\lambda I + G'(\bar{u}))^{-1}\|^2_{\mathcal{L}(L^2(\Omega))}\|v\|^2_{L^2(\Omega)} = \tilde{\delta}\|v\|^2_{L^2(\Omega)} \end{aligned}$$

holds for all $v \in \widehat{C}_\tau$ with $\tilde{\delta} > 0$.

In particular, we take $v = \lambda u + G(u)$ with $y_a(x) \leq \lambda u(x) + G(u)(x) \leq y_b(x)$ and $\|u - \bar{u}\|_{L^\infty(\Omega)} \leq \epsilon$ such that $\|v - \bar{v}\|_{L^\infty(\Omega)} \leq \tilde{\epsilon}$ and $\|v - \bar{v}\|_{L^2(\Omega)} \leq \rho_1$. Because of (R), to every $v \in V_{ad}$ with $\|v - \bar{v}\|_{L^2(\Omega)} \leq \rho_1$ with $u = K(v)$ and $\|u - \bar{u}\|_{L^2(\Omega)} \leq r_1$.

From continuity of the mapping $\lambda I + G$ from $L^\infty(\Omega)$ to $L^\infty(\Omega)$ implies that $\|u - \bar{u}\|_{L^\infty(\Omega)} \leq \epsilon$, hence $\|v - \bar{v}\|_{L^\infty(\Omega)} \leq r$. If ϵ is sufficiently small, then $r \leq \tilde{\epsilon}$ and $\|v - \bar{v}\|_{L^2(\Omega)} \leq c\|v - \bar{v}\|_{L^\infty(\Omega)} \leq \rho_1$.

Since $\|u - \bar{u}\|_{L^\infty(\Omega)} \leq \epsilon$, so $\|v - \bar{v}\|_{L^2(\Omega)} \leq \tilde{\epsilon}$. Then, with $F(v) = f(u)$ and $F(\bar{v}) = f(\bar{u})$. From the growth condition (6.3) we have

$$f(u) \geq f(\bar{u}) + \hat{\sigma}\|\lambda u + G(u) - (\lambda\bar{u} + G(\bar{u}))\|^2_{L^2(\Omega)} \tag{6.9}$$

for all u with $\lambda u + G(u) \in V_{ad}$ and $\|u - \bar{u}\|_{L^\infty(\Omega)} \leq \epsilon$.

This implies the local optimality of $\bar{u}$. We show that the quadratic growth condition (6.7) A Taylor expansion for the last term in (6.9)

$$\lambda u + G(u) - (\lambda \bar{u} + G(\bar{u})) = \lambda(u - \bar{u}) + G'(u)(u - \bar{u}) + r_1^G(\bar{u}, u - \bar{u})$$

and since G is continuously Frechet differentiable from $L^2(\Omega)$ to $W^\infty\{a_\alpha, 2\} \cap C(\bar{\Omega})$, the remainder term satisfies

$$\frac{\|r_1^G\|_{L^2(\Omega)}}{\|u - \bar{u}\|_{L^2(\Omega)}} \to 0 \qquad as \qquad \|u - \bar{u}\|_{L^2(\Omega)} \to 0 \tag{6.10}$$

Therefore, we obtain

$$\begin{aligned}
\|\lambda u + G(u) - (\lambda \bar{u} + G(\bar{u}))\|_{L^2(\Omega)} &= \|(\lambda I + G'(\bar{u}))(u - \bar{u}) + r_1^G\|_{L^2(\Omega)} \\
&\geq \|(\lambda I + G'(\bar{u}))(u - \bar{u})\|_{L^2(\Omega)} - \|r_1^G\|_{L^2(\Omega)} \\
&\geq \left(\frac{1}{\|(\lambda I + G'(\bar{u}))^{-1}\|_{\mathcal{L}(L^2(\Omega))}} - \frac{\|r_1^G\|_{L^2(\Omega)}}{\|u - \bar{u}\|_{L^2(\Omega)}} \right) \|u - \bar{u}\|_{L^2(\Omega)} \\
&\geq \hat{c}\|u - \bar{u}\|_{L^2(\Omega)}
\end{aligned}$$

Since $(\lambda I + G'(\bar{u}))$ is continuously invertible because of (R), (6.10) yields

$$f(u) \geq f(\bar{u}) + \hat{\sigma}\hat{c}^2\|u - \bar{u}\|_{L^2(\Omega)} = f(\bar{u}) + \sigma\|u - \bar{u}\|_{L^2(\Omega)}$$

□

References

[1] T. Bayen, J. F. Bonnans, and F. J. Silva, *Characterization of local quadratic growth for strong minima in the optimal control of semilinear elliptic equation*, Trans. Amer. math. Soc. **366** (2013), 2063–2087.

[2] J. F. Bonnans, *Second-order analysis for control constrained optimal control problems of semilinear elliptic systems*, Appl. Math. Optim. **38** (1998), 303–325.

[3] E. Casas, *Control of an elliptic problem with pointwise state constraints*, SIAM J. Control and Optimization **4** (1986), 1309–1322.

[4] E. Casas and M. Mateos, *second order optimality conditions for semilinear elliptic control problems with finitely many state constraints*, SIAMJ. Control. Optim. **40** (2002), 1431–1454.

[5] E. Casas, J. C. D. L. Reyes, and F. Tröltzsch, *Sufficient second-order optimality conditions for semilinear control problems with pointwise state constraints*, SIAMJ. Optim. **19** (2008), 616–643.

[6] E. Casas and F. Tröltzsch, *Second order necessary and sufficient optimality conditions for optimization problems and applications to control theory*, SIAMJ. Control. Optim. **13** (202), 406–431.

[7] E. Casas, F. Tröltzsch, and A. Unger, *Second order sufficient optimality conditions for some state-constrained control problems of semilinear equations*, SIAM J. Control and Optimization **38** (2000), no. 5, 1369–1391.

[8] A. Ju. Dubinskii, *Sobolev space of infinite order and the behavior of solutions of some boundeary value problems with unbounded increase of the order of the equation*, Math. USSR Sbornik **27** (1975), no. 2.

[9] ______, *Some imbeding theorem for sobolev spaces of infinite order*, Soviet Math. Dokl. **19** (1979), 1271–1274.

[10] S. A. El-Zahaby, *Second order sufficient optimality conditions for semilinear elliptic control problem of infinite order*, Accepted for Publication in International Journal of Pure and Applied Mathematical.

[11] S. A. El-Zahaby and Samira El-Tamimy, *Optimal control problems of systems governed by elliptic operator of infinite order with pointwise control constraints*, Accepted for publication in Journal of Mathematics and Statistical Science (ISSN2411-2518, USA) **DEcember 25** (2015).

[12] S. A. El-Zahaby and GH. H. Mostafa, *Optimal control of variational inequalities of infinite order*, J. of Math. Egyptian Soc. **13** (2005), 99–106.

[13] I. M. Gali, *Optimal control of a system governed by elliptic operator of inifinte order ordinary and partial differential-equations*, Proceedings Dunder Scolled, Springer-Verlag series, Lecture Notes in Mathematics **964** (1982), 263–272.

[14] I. M. Gali and H. A. El-Saify, *Distributed control of systems coverned by dirichlet and neumann problems for a self-adjoint elliptic operator with an infinite number of variables*, J. of Optimization Theory and Applications **39** (1983), no. February, 293–298.

[15] ______, *Optimal control of a system governed by a self adjoint elliptic operator with infinite number of variable*, Proceeding of international conference on "Functional Differential Systems and Related Topics" **Poland** (May, 1991).

[16] I. M. Gali, H. A. El-Saify, and El-Zahaby, *Optimal control of a system governed by hyperbolic operator*, Springer-Verlg Series, Lecher Notes in Math. Proceedings at the Eighth Conference of The Ordinary and Optimal Differential Equations Hold at Dundee, Scotland, Apper in Lecture Notes in Math Springer-Verlg Series **June** (1984), 26–29.

[17] ______, *Distributed control of a system governed by dirichlet and neumann problem for elliptic equations of inifite order*, The international conference in Functional-Differential Systems and Related Topics **III** (May 1983), Blazejewko, Poland.

[18] I. M. Gali and El-Zahaby, *Optimal control of a system governed by variational inequalities for infinte order elliptuc operator*, Kyngpook Mathematical Journal **31** (December 1991), no. 2, 193–200.

[19] C. Meyer and F. Tröltzsch, *On an elliptic optimal control probelm with pointwise mixed control-state constraints*, In: Seeger, A. (ed). Recent Advance in optimization, of Lecture Notes in Economics and Mathemtical Systems, Springer Berlin, Heidelberg **653** (2006), 187–204.

[20] A. Röch and F. Tröltzsch, *Existence of regular lagrange multipliers for elliptic optimal control with pointwise control-state constraints*, preprint (2005a).

[21] F. Tröltzsch, *Optimal control of partial differential equation: Theory, methods and applications*, American Mathematical Society Provndence, Rhode, Island (2010).