On modified Soft rough sets on a complete atomic Boolean lattice

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Abstract

Soft set theory and rough set theory are treated as mathematical approaches to deal with uncertainty. They are combined together by F.Feng et al. In[29], we introduce the notion of soft rough sets on a complete atomic Boolean lattice as a generalization of soft rough sets. In this paper, we strengthen the concept of soft rough on a complete atomic Boolean lattice by defining the concept of MSR sets on a complete atomic Boolean lattice. In this model, some properties which were not satisfied in soft rough sets on a complete atomic Boolean lattice can be proved. Finally, the notion of approximations of Boolean lattice information system with to another Boolean lattice information system is studied and we gave an applicable example to illustrate this notion.

Keywords: Complete atomic Boolean lattice; soft rough approximation operators on a complete atomic Boolean lattice- MSR sets on a complete atomic Boolean lattice.

1. Introduction

In recent years, scientists, engineers and mathematicians have shown great interest in uncertainty as it found many fields like decision making, engineering, environmental science, social sciences, and medical science etc. Probability theory, fuzzy set theory [1], rough set theory [2, 3] and other mathematical tools have been used successfully to describe uncertainty. Each of these ideas have its inherent difficulties as pointed out in [4,5]. Consequently, Molodtsov proposed a novel concept for modelling vagueness and uncertainty called soft set theory.

Theory of soft sets has enough parameters, so that it is free from above mentioned difficulties. It deals with uncertainty and vagueness on the one hand while on the other it has enough parametrization tools. These qualities of soft set theory make it popular among researchers and experts working in diverse areas. Applications of soft set theory can be seen in [6-12] Research on soft set theory is growing rapidly [13-17]. Maji and Roy [17], Maji et al. [18,19] further studied soft set theory and used it to solve some decision making problems for fuzzy soft sets by combining soft sets with fuzzy sets. Roy and Maji [20] presented a fuzzy soft set theoretic approach to decision making problems. Jiang et al. [21] extended soft sets with description logic. Aktas and Cagman [22] defined soft groups. Feng et al. [23,24] investigated relationships among soft sets, rough sets and fuzzy sets. Shabir and Naz [25] investigated soft topological spaces. Ge et al. [26] discussed relationships between soft sets and topological spaces. Li and Xie [27] studied the relationships among soft sets, soft rough sets and topologies. Ali [28] introduced the concept of a soft binary relation and investigated the soft upper and lower approximation operations with respect to soft equivalence relations. In [29] we presented soft sets on a complete atomic Boolean lattice as a generalization of soft sets and obtained the lattice structure of these soft sets. Li et al. [30] considered soft coverings.

Rough set theory, introduced by Pawlak [31] in the 1980s, is a powerful machine learning tool that has applications in many data mining [32-34] instances, attribute and feature selection [35-37], and data prediction [38, 39]. Rough set theory was proposed as a formal tool for modeling and processing incomplete information in information system. Pawlak rough set is mainly based on equivalence relation. But in practical it is very difficult to find an equivalence relation among the element of a set. So, some other general relations such as tolerance one and dominance ones are considered to define rough set models. Many interesting and meaningful extensions of Pawlak rough sets have been presented in the literature. Equivalence relations can be replaced by tolerance relations [40], similarity relations[41] and binary relations [42-44]. The properties of the rough approximations in more general setting of complete atomic Boolean lattice were studied by Järvinen in [45]. All these proposals share the common feature that they deal with approximations of concepts in terms of granules.

The major criticism on rough set theory is that it lacks parametrization tools. In order to make parametrization tools available in rough sets a major step is taken by Feng et al. in [46]. They introduce the concept of soft rough sets, where instead of equivalence classes parameterized subsets of a set serve the purpose of finding lower and upper approximations of a subset. Also, soft sets are combined with fuzzy sets and rough sets in [47]. The concept of rough soft group is defined in [48]. In [49], Shabir introduced a new approach to soft rough sets called modified soft rough set (MSR-set) and studied some of their basic properties. In [29], we defined two pairs of soft rough approximation operators on a complete atomic Boolean lattice and give their properties. These operators suffer from unexpected properties such as soft upper approximation of non zero element might be equal zero and soft upper approximation of any element might not greater than this element. To resolve this problem, we introduce the notion of modified soft rough set on a complete atomic Boolean lattice and its application in decision making was analyzed.

This paper is arranged as follows, in section 2, some basic concepts of soft sets and rough sets on a complete atomic Boolean lattice are discussed. Also, we discuss the notion of soft rough set on a complete atomic Boolean lattice. The purpose of section is to introduce the notion of modified soft rough sets on a complete atomic Boolean lattice and study their properties. In section 4, we introduce the notion of Boolean lattice information system. Also, the relationship between the Boolean lattice information and soft sets on a complete atomic Boolean lattice is discussed. Also, We study the relation between the mapping induced by a Boolean lattice information system induced by a soft set f_A on a complete atomic Boolean lattice B and the mapping induced by f_A on B. In the last section, the concept of approximations of Boolean lattice information system with respect to another Boolean lattice information system is studied. Finally we introduce an applicable example to illustrate this notion.

2. Preliminaries

We assume that the reader is familiar with the usual lattice-theoretical notation and conventions, which can be found in [50, 51].

First we recall some definitions and properties of maps. Let $\mathbf{B} = (B, \leq)$ be an ordered set. A mapping $f : B \longrightarrow B$ is said to be extensive, if $x \leq f(x)$ for all $x \in B$. The map f is order preserving if $x \leq y$ implies $f(x) \leq f(y)$. Moreover, f is idempotent if $f(f(x)) = f(x)$ for all $x \in B$. A map $c : B \longrightarrow B$ is said to be a closure operator on B, if c is extensive, order-preserving, and idempotent. An element $x \in B$ is c-closed if c(x)=x. Furthermore, if $i : B \longrightarrow B$ is a closure operator on $\mathbf{B}^{\vartheta} = (B, \geq)$ then i is an interior operator on B. Let $\mathbf{B} = (B, \leq)$ and $\mathbf{Q} = (Q, \leq)$ be ordered sets. $f : B \longrightarrow Q$ is an order embedding, if for any $a, b \in B$, $a \leq b$ in B if and only if $f(a) \leq f(b)$ in Q, note that an order embedding is always an injection. An order-embedding f onto Q is called an order-isomorphism between $\mathbf{B}$ and $\mathbf{Q}$, we say that $\mathbf{B}$ and $\mathbf{Q}$ are order-isomorphic and write $\mathbf{B} \cong \mathbf{Q}$. If $\mathbf{B} = (B, \leq)$ and $\mathbf{Q} = (Q, \leq)$ are order-isomorphic, then $\mathbf{B}$ and $\mathbf{Q}$ are said to be dually order-isomorphic. A pair $(^{\triangledown}, ^{\vartriangle})$ of maps $^{\triangledown} : B \longrightarrow B$ and $^{\vartriangle} : B \longrightarrow B$ is called a dual Galois connection on B if $^{\triangledown}$ and $^{\vartriangle}$ are order preserving and $x^{\triangledown\vartriangle} \leq x \leq x^{\vartriangle\triangledown}$ for all $x \in B$.

Before we consider the Boolean lattices, we present the following lemma, where $\wp(B)$ denotes the power set of B, that is, the set of all subsets of B.

Lemma 2.1 [50] Let $\mathbf{B} = (B, \leq)$ be a complete lattice, S, $T \subseteq B$ and $\{X_i : i \in I\} \subseteq \wp(B)$

i) If $S \subseteq T$, then $\bigvee S \subseteq \bigvee T$.

ii) $\bigvee (S \cup T) = (\bigvee S) \vee (\bigvee T)$.

iii) $\bigvee (\bigcup \{X_i : i \in I\}) = \bigvee \{\bigvee X_i \in I\}$.

Next we recall the concept of Boolean lattices. They are bounded distributive lattices with a complementation operation.

Definition 2.2 [50] A lattice $\mathbf{B} = (B, \leq)$ is called a Boolean lattice, if

i) B is distributive,

ii) B has a least element 0 and a greatest element 1, and

iii) Each $x \in B$ has a complement $x' \in B$ such that $x \vee x' = 1$ and $x \wedge x' = 0$.

Lemma 2.3 [50] Let $\mathbf{B} = (B, \leq)$ be a Boolean lattice, then for all $x,\ y \in B$

i) $0' = 1$ and $1' = 0$,

ii) $x'' = x$,

iii) $(x \vee y)' = x' \wedge y'$, and $(x \wedge y)' = x' \vee y'$,

iv) $x \leq y$ iff $x \wedge y' = 0$.

Lemma 2.4 [50] Let $\mathbf{B} = (B, \leq)$ be a complete Boolean lattice. Then for all $\{x_i : i \in I\} \subseteq B$ and $y \in B$

$$y \wedge (\bigvee_{i \in I} x_i) = \bigvee_{i \in I} (y \wedge x_i)$$

and

$$y \vee (\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} (y \vee x_i)$$

Definition 2.5 [45] Let $\mathbf{B} = (B, \leq)$ be an ordered set and $x,\ y \in B$, we say that x is covered by y (or that y covers x), and write, $x \prec y$ if $x < y$ and there is no element z in B with $x < z < y$.

Definition 2.6 [45] Let $\mathbf{B} = (B, \leq)$ be a lattice with a least element 0. Then $a \in B$ is called an atom if $0 \prec a$. The set of atoms of B is denoted by $A(B)$. The lattice B is called atomic if every element of B is the supremum of the atoms below it, that is $x = \bigvee \{a \in A(B) : a \leq x\}$.

It is obvious that in a lattice $\mathbf{B} = (B, \leq)$ with a least element 0,

$$a \wedge x \neq 0 \Leftrightarrow a \leq x$$

for all $a \in A(B)$ and $x \in B$. This implies that $a \wedge b = 0$ for all $a, b \in A(B)$ s.t $a \neq b$. Furthermore, if B is atomic, then for all $x \neq 0$ there exists an atom $a \in A(B)$ s.t $a \leq x$. Namely, if $\{a \in A(B) : a \leq x\} = \phi$, then $x = \bigvee \{a \in A(B) : a \leq x\} = \bigvee \phi = 0$.

Definition 2.7 [45] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\psi : A(B) \to B$ be any mapping. For any element $x \in B$, let

$x^{\nabla} = \bigvee\{a \in A(B) : \psi(a) \le x\}$, and

$x^{\triangle} = \bigvee\{a \in A(B) : \psi(a) \wedge x \neq 0\}$.

The elements x^{∇} and $x^{\triangle}$ are called the **lower** and the **upper** approximations of x with respect to ψ respectively. Two elements x and y are called equivalent if they have the same upper and lower approximations. The resulting equivalence classes are called rough sets.

Definition 2.8[29] Let $\mathbf{B} = (B, \le)$ be a complete atomic Boolean lattice and E be a set of parameters. Let A be a non empty subset of E . A soft set over A, with support A , denoted by f_A on B is defined by the set of ordered pairs

$$f_A = \{(e, f_A(e)) : e \in E, f_A(e) \in B\},$$

or is a function $F_A : E \to B$ s.t

$$f_A(e) \neq 0 \quad \forall \quad e \in A \subseteq E \, and \, f_A(e) = 0 \, if \; e \notin A.$$

In other words, a soft set over B is a parameterized family of elements of B. For each $e \in A$, $f(e)$ is considered as e-approximate element of f_A

Definition 2.9[29] Let $\mathbf{B} = (B, \le)$ be a complete atomic Boolean lattice. Let $A_1, A_2 \subseteq E$ and let f_{A_1} and g_{A_2} be two soft sets over B.

i) f_{A_1} is a soft subset of g_{A_2}, denoted by $f_{A_1} \sqsubseteq g_{A_2}$ if $A_1 \subseteq A_2$ and $f(e) \le g(e)$ for every $e \in A_1$.

ii) f_{A_1} and g_{A_2} are called soft equal, denoted by $f_{A_1} = g_{A_2}$ if $f_{A_1} \sqsubseteq g_{A_2}$ and $g_{A_2} \sqsubseteq f_{A_1}$.

Definition 2.10 [29] Let $\mathbf{B} = (B, \le)$ be a complete atomic Boolean lattice. Let $A \subseteq E$ and let f_A be a soft set over B.

i) f_A is called null, denoted by 0_A if $f(e) = 0$ for every $e \in A$.

ii) f_A is called absolute, denoted by 1_A if $f(e) = 1$ for every $e \in A$.

We stipulate that 0_{ϕ} is also a soft set over B with $0 : \phi \to B$.
Let $A \subseteq E$ and let f_A be a soft set over B. Obviously,

$$0_A \sqsubseteq f_A \sqsubseteq 1_A$$

Below, we introduce some operations on soft sets on B and investigate their properties.

Definition 2.11[29] Let $\mathbf{B} = (B, \le)$ be a complete atomic Boolean lattice. Let $A_1, A_2 \subseteq E$ and let f_{A_1} and g_{A_2} be two soft sets over B.

i) h_{A_3} is called the intersection of f_{A_1} and g_{A_2}, denoted by $f_{A_1} \sqcap g_{A_2} = h_{A_3}$ if $A_3 = A_1 \cap A_2$ and $h(e) = f(e) \wedge g(e)$ for every $e \in A_3$.

ii) h_{A_3} is called the union of f_{A_1} and g_{A_2}, denoted by $f_{A_1} \sqcup g_{A_2} = h_{A_3}$ if $A_3 = A_1 \cup A_2$ and $h(e) = f(e)$ if $e \in A_1 - A_2$, $h(e) = g(e)$ if $e \in A_2 - A_1$ and $h(e) = f(e) \vee g(e)$ if $e \in A_1 \cap A_2$.

Definition 2.12 [29] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice. Let $A \subseteq E$ and let f_A be a soft set over B. The complement of f_A, denoted by $(f_A)^c$ is defined by $(f_A)^c = (f^c, A)$, where $f^c : A \to B$ is a mapping given by $f^c(e) = f(e)'$ for every $e \in A$.

Definition 2.13[29] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and Let f_E be a soft set over B.

i) f_E is called full, if $\bigvee_{e \in E} f(e) = 1$;

ii) f_E is keeping infimum, if for any $e_1, e_2 \in E$, there exists $e_3 \in E$ such that $f(e_1) \wedge f(e_2) = f(e_3)$;

iii) f_E is keeping supremum, if for any $e_1, e_2 \in E$, there exists $e_3 \in E$ such that $f(e_1) \vee f(e_2) = f(e_3)$;

iv) f_E is called partition of B if

- **1)** $\bigvee_{e \in E} f(e) = 1$,
- **2)** For every $e \in E$, $f(e) \neq 0$,
- **3)** For every $e_1, e_2 \in E$ either $f(e_1) = f(e_2)$ or $f(e_1) \wedge f(e_2) = 0$.

Obviously, every partition soft set is full and f_E is keeping infimum(resp. keeping supremum) if and only if for every $E^* \subseteq E$, there exists $e^* \in E$ such that $\bigwedge_{e \in E^*} f(e) = f(e^*)$(resp.$\bigvee_{e \in E^*} f(e) = f(e^*)$).

Definition 2.14[29] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. For any element $x \in B$, we define a pair of operators $x^\vee, x^\wedge : B \to B$ as follows:

$$x^\vee = \bigvee \{b \in A(B) : \exists e \in A \; s.t \; b \leq f(e) \; and \; f(e) \leq x\},$$

$$x^\wedge = \bigvee \{b \in A(B) : \exists e \in A \; s.t \; b \leq f(e) \; and \; f(e) \wedge x \neq 0\}.$$

The elements $x^\vee$ and $x^\wedge$ are called the **soft lower** and the **soft upper** approximations of x over B. Two elements x and y are called soft equivalent if they have the same soft upper and soft lower approximations over B. The resulting equivalence classes are called soft rough sets over B.

Proposition 2.15 [29] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Then the following properties hold.

1) If f_A is full, then

> **i)** $x^\vee \leq x \leq x^\wedge$;
>
> **ii)** $1^\vee = 1^\wedge = 1$.

2) If f_A is keeping supremum, then

> **i)** For all $x \in B$, $\exists e \in A,\ s.t\, x^\vee = f(e)$;
>
> **ii)** For all $x \in B$, $\exists e \in A,\ s.t\, x^\wedge = f(e)$.

3) If f_A is full and keeping supremum, then

> $x^\wedge = 1$ for every $x \in B$ and $x \neq 0$.

Example 2.16 Let $B = \{0, a, b, c, d, e, f, 1\}$ and let the order $\leq$ be defined as in figure 1.

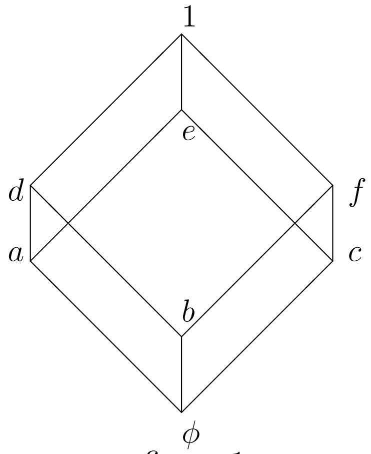

figure 1

The set of atoms of a complete atomic Boolean lattice $\mathbf{B} = (B, \leq)$ is {a,b,c}. Let $A = \{e_1, e_2, e_3, \}$ and f_A be a soft set over B defined as follows:
$f(e_1) = a$, $f(e_2) = b$, and $f(e_3) = d$. So f_A is not full. Many odd situation accurs. For example,
If $x = c \neq 0$, then $x^\vee = x^\wedge = 0$. Also if if $x = e$, then $x^\wedge = a \vee b = d \ngeq e$ Morover $c \nleq x^\vee$ or $c \nleq x^\wedge$ for any $x \in B$
In order to avoid these situations, we intoduced in [29] the notion of full soft sets on B. Morover the concept of soft postive, soft negative and soft boundary are meaingful in the case of full soft sets.

In the following, we show that negative element of any $x \in B$ B cannot be avoided.

Proposition 2.17 Let f_A be a soft set over B which is not full. Then there exists at least $b \in A(B)$ such that $b \leq neg(x) = (x^\wedge)'$ for all $x \in B$.
Proof: Since f_A is not full, i.e. $\bigvee_{a \in A} f(a) \nleq 1$. So $\exists b \in A(B)\ s.t\ b \nleq f(a)\ \forall\ a \in A$. Let $x \in B$ s.t $b \leq x$. If $b \leq x^\wedge$, then $\exists a \in A\ s.t\ b \leq f(a)$ and $f(a) \wedge x \neq 0$ which is a contradiction. Hence $b \nleq x^\wedge$. By a similar argument when $b \nleq x$ it can be shown that $b \nleq x^\wedge$. Therefore $b \leq 1 - x^\wedge = (x^\wedge)'$.

3. Modified Soft Rough sets (MSR-sets) on a Complete atomic Boolean Lattice

In the following we strengthen the concept of soft rough sets on a complete atomic Boolean lattice B by defining modified soft rough sets on B.

Definition 3.1 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $\varphi : A(B) \to \wp(A)$ be another map defined as $\varphi(b) = \{a \in A : b \leq f(a)\}$. Then the pair $(A(B), \varphi)$ is called MSR-approximation space on B and for any element $x \in B$, lower MSR-approximation space on B is defined as

$$x_\varphi^\vee = \bigvee \{a \in A(B) : a \leq x\ , \varphi(a) \neq \varphi(b)\ \forall\, b \in A(B)\ s.t\ b \nleq x\},$$

and its upper MSR-approximation over B is defined as

$$x_\varphi^\wedge = \bigvee \{a \in A(B) : \varphi(a) = \varphi(b)\ for\ some\ b \in A(b)\ s.t\ b \leq x\}.$$

If $x_\varphi^\vee \neq x_\varphi^\wedge$, Then x is said to be MSR-element over B.

Remark 3.2 Lower MSR-approximation of x over B can be defined as $x_\varphi^\vee = \bigvee \{a \in A(B) : \varphi(a) \neq \varphi(b)\ \forall\, b \in A(B)\ s.t\ b \nleq x\}$ because if we let $\varphi(a) \neq \varphi(b)\ \forall\ b \in A(B)$ s.t $b \nleq x$ and $a \nleq x$, then by hypothesis $\varphi(a) \neq \varphi(a)$ which is impossible.

Lemma 3.3 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space. Then for all $c \in A(B)$ and $x \in B$

i) $c \leq x_\varphi^\vee \Longleftrightarrow c \leq x$ and $\varphi(c) \neq \varphi(b)\ \forall\, b \in A(B)\ s.t\ b \nleq x$;

ii) $c \leq x_\varphi^\wedge \Longleftrightarrow \varphi(c) = \varphi(b)$ for some $b \in A(B)$ s.t $b \leq x$.

Proof: **(i)**($\Rightarrow$) Suppose that $c \leq x_\varphi^\vee = \bigvee \{a \in A(B) : a \leq x\ , \varphi(a) \neq \varphi(b)\ \forall\, b \in A(B)\ s.t\ b \nleq x\}$. So $c \leq x$. If $\varphi(c) \neq \varphi(b)\ \forall\, b \in A(B)$ s.t $b \nleq x$, then $c \wedge x_\varphi^\vee = c \wedge \bigvee \{a \in A(B) : a \leq x\ , \varphi(a) \neq \varphi(b)$

$\forall\, b \in A(B)\; s.t\; b \not\leq x\}$= $\bigvee\{a \wedge c : a \in A(B), c \leq x\;, \varphi(a) \neq \varphi(b)\;\forall\, b \in A(B)\; s.t\; b \not\leq x\}$. Since $\varphi(c) \neq \varphi(b)$, then $c \neq a$, i.e. $c \wedge a = 0$. Hence $c \leq (x^{\vee}_{\varphi})'$, a contradiction.
($\Leftarrow$) Suppose that $c \leq x$ and $\varphi(c) \neq \varphi(b)\;\forall\, b \in A(B)\; s.t\; b \not\leq x$, then $c \leq \bigvee\{a \in A(B) : a \leq x\;, \varphi(a) \neq \varphi(b)\;\forall\, b \not\leq x\} = x^{\vee}_{\varphi}$.
Condition(ii) can be proved similarly.

The following proposition shows a relation between soft lower approximation operators over a complete a complete atomic Boolean lattice B and lower MSR-approximation over B.

Proposition 3.4 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space. Then for any $x \in B$, $x^{\vee} \leq x^{\vee}_{\varphi}$.
Proof: Let $b \in A(B)$ s.t $b \leq x^{\vee}$. Then $\exists\, a \in A\; s.t\; b \leq f(a) \leq x$. So $a \in \varphi(b)$ and $b \leq x$. If $b \not\leq x^{\vee}_{\varphi}$, then $\varphi(b) = \varphi(c)$ for $c \in A(B)$ s.t $c \not\leq x$. Since $a \in \varphi(b)$ and $\varphi(b) = \varphi(c)$, then $a \in \varphi(c)$ which implies that $c \leq f(a) \leq x$. Hence $c \leq x$ which is a contradiction. Therefore $x^{\vee} \leq x^{\vee}_{\varphi}$.

The following example shows that the relation $\leq$ between $x^{\vee}_{\varphi}$ and $x^{\vee}$ may be proper.

Example 3.5 Let $B = \{0, a, b, c, d, e, f, 1\}$ and let the order $\leq$ be defined as in figure 1. Let $A = \{e_1, e_2, e_3, \}$ and f_A be a soft set over B defined as follows:
$f(e_1) = a$, $f(e_2) = 0$, and $f(e_3) = e$. Then the map φ of MSR-approximation space $(A(B), \varphi)$ will be $\varphi(a) = \{e_1, e_3\}$, $\varphi(b) = \phi$, and $\varphi(c) = \{e_3\}$.
Let $x = d$. Then $x^{\vee}_{\varphi} = a \vee b = d$ and $x^{\vee} = a$. Thus $x^{\vee} \leq x^{\vee}_{\varphi}$.

In the following proposition we study a necessary and sufficient condition for $x^{\vee}_{\varphi} \leq x^{\vee}$ to be hold for any $x \in B$.

Proposition 3.6 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space. Then for any $x \in B$, $x^{\vee}_{\varphi} \leq x^{\vee}$ iff for every $b \in A(B)\;\exists\; e \in A$ s.t $f(e) = \bigvee\{a \in A(B) : \varphi(a) = \varphi(b)\}$.
Proof: ($\Rightarrow$) Assume that $x^{\vee}_{\varphi} \leq x^{\vee}$ for any $x \in B$. Let $b \in A(B)$ and $x = \bigvee\{a \in A(B) : \varphi(a) = \varphi(b)\}$. So $x^{\vee}_{\varphi} = \bigvee\{a \in A(B) : \varphi(a) = \varphi(c)\; for\; some\; c \in A(B)\; c \leq x\}$ $= \bigvee\{a \in A(B) : \varphi(a) = \varphi(b)\} = x$. Since $b \leq x = x^{\vee}_{\varphi}$ and $x^{\vee}_{\varphi} \leq x^{\vee}$, then $b \leq x^{\vee}$. Therefore $e \in A$ s.t $b \leq f(e)$ and $f(e) \leq x$. Also, for all $a \in A(B)$ s.t $a \leq x$, we have $\varphi(a) = \varphi(b)$,thus $e \in \varphi(b) = \varphi(a)$. So $a \leq f(e)$ and therefore $x \leq f(e)$. Consequently, $f(e) = x = \bigvee\{a \in A(B) : \varphi(a) = \varphi(b)\}$.
($\Leftarrow$) Let $x \in B$ and $b \in A(B)$ s.t $b \leq x^{\vee}_{\varphi}$. So for every $a \in A(B)$, if $\varphi(ba) = \varphi(b)$, then $a \leq x$. Hence $\bigvee\{a \in A(B) : \varphi(a) = \varphi(b)\} \leq x$. By assumption, $\exists\; e \in A$ s.t $f(e) = \bigvee\{a \in A(B) : \varphi(a) = \varphi(b)\}$. Hence $b \leq f(e)$ and $f(e) \leq x$. Therefore $b \leq x^{\vee}$ and

consequently, $x_\varphi^\vee \leq x^\vee$.

Remark 3.7 In general, there is no relation between $x_\varphi^\wedge$ and $x^\wedge$. If $x = e$ in Example 2.10, then $x^\wedge = d$ and $x_\varphi^\wedge = e \not\leq d$.

In Proposition 3.8, 3.9 and 3.10, we show that there is a relation between $x_\varphi^\wedge$ and $x^\wedge$ if some specific conditions hold.

Proposition 3.8 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space. Then f_A is full iff $x_\varphi^\wedge \leq x^\wedge$ for every $x \in B$.
Proof: ($\Rightarrow$) Assume that f_A is full and $x \in B$. Let $a \in A(B)$ s.t $a \leq x_\varphi^\wedge$, then $\exists\ b \in A(B)$ s.t $b \leq x$ and $\varphi(a) = \varphi(b)$. Since $b \leq 1 = \bigvee_{e \in A} f(e)$, then $\exists\ e \in A$ s.t $b \leq f(e)$. Hence $b \leq f(e) \wedge x$ and thus $f(e) \wedge x \neq 0$. By, $b \leq f(e)$, we have $e \in \varphi(b) = \varphi(a)$ and thus $a \leq f(e)$. Consequently, $x_\varphi^\wedge \leq x^\wedge$.
($\Leftarrow$) Suppose that $x_\varphi^\wedge \leq x^\wedge$ for every $x \in B$, we show that $1 \leq \bigvee_{e \in A} f(e)$. Let $a \in A(B)$, then $a \leq a_\varphi^\wedge \leq a^\wedge$. Therefore $\exists\ e \in A$ s.t $f(e) \wedge a \neq 0$ and thus $a \leq f(e)$ because $a \in A(B)$. Consequently, f_A is a full soft set over B.

Proposition 3.9 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space. Then $x^\wedge \leq x_\varphi^\wedge$ for every $x \in B$ iff for every $e_1, e_2 \in A$, $f(e_1) \wedge f(e_2) = 0$ whenever $f(e_1) \neq f(e_2)$.
Proof: ($\Rightarrow$) Assume that $x^\wedge \leq x_\varphi^\wedge$ for every $x \in B$. Let $e_1, e_2 \in A$, if $f(e_1) \wedge f(e_2) \neq 0$, then $\exists\ b \in A(B)$ s.t $b \leq f(e_1) \wedge f(e_2)$. Since $b \leq f(e_1)$, then $f(e_1) \leq \bigvee\{f(e) : b \leq f(e)\} = \bigvee\{f(e) : b \wedge f(e) \neq 0\} = b^\wedge \leq b_\varphi^\wedge = \bigvee\{a \in A(B) : \varphi(a) = \varphi(b)\}$. On the other hand we show that $\bigvee\{a \in A(B) : \varphi(a) = \varphi(b)\} \leq f(e_1)$. Let $c \in A(B)$ s.t $c \leq \bigvee\{a \in A(B) : \varphi(a) = \varphi(b)\}$, then $\varphi(c) = \varphi(b)$ and thus $e_1 \in \varphi(b) = \varphi(c)$. Therefore $c \leq f(e_1)$ and consequently, $f(e_1) = \bigvee\{a \in A(B) : \varphi(a) = \varphi(b)\}$. Similarly, by $b \leq f(e_2)$, $f(e_2) = \bigvee\{a \in A(B) : \varphi(a) = \varphi(b)\}$ and hence $f(e_1) = f(e_2)$.
($\Leftarrow$) Assume that for every $e_1, e_2 \in A$, $f(e_1) \wedge f(e_2) = 0$ whenever $f(e_1) \neq f(e_2)$. Let $x \in B$ and $a \in A(B)$ s.t $a \leq x^\wedge$, then $\exists\ e_1 \in A$ s.t $a \leq f(e_1)$ and $f(e_1) \wedge x \neq 0$. So, $\exists\ b \in A(B)$ s.t $b \leq f(e_1) \wedge x$. We show that $\varphi(a) = \{e_2 \in A : f(e_2) = f(e_1)\}$. If $f(e_2) \neq f(e_1)$, then $f(e_1) \wedge f(e_2) = 0$ by assumption. Thus $a \not\leq f(e_2)$ because $a \leq f(e_1)$ and therefore $e_2 \notin \varphi(a)$. So $\varphi(a) \subseteq \{e_2 \in A : f(e_2) = f(e_1)\}$. On the other hand, if $f(e_2) = f(e_1)$, then $a \leq f(e_2)$ and hence $e_2 \in \varphi(a)$. Consequently, $\varphi(a) = \{e_2 \in A : f(e_2) = f(e_1)\}$. Similarly we can show that $\varphi(b) = \{e_2 \in A : f(e_2) = f(e_1)\}$ and thus $\varphi(a) = \varphi(b)$. Since $b \leq x$, then $a \leq x_\varphi^\wedge$.

Corollary 3.10 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space. If $f(e) \neq 0$ for every $e \in A$, then f_A is a partition soft set iff $x^\wedge = x_\varphi^\wedge$ for every $x \in B$.
Proof: Obvious.

Proposition 3.11 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space. If $\varphi(a) = \varphi(b)$ for some $a, b \in A(B)$, then for any $x \in B$ either $a, b \leq x^{\wedge}_{\varphi}$ or $a, b \nleq x^{\wedge}_{\varphi}$.

Proof: If $a \leq x^{\wedge}_{\varphi}$, then $\varphi(a) = \varphi(c)$ for some $c \in A(B)$ s.t $c \leq x$. Since $\varphi(a) = \varphi(b)$, then $\varphi(b) = \varphi(c)$, $c \leq x$. This implies that $b \leq x^{\wedge}_{\varphi}$.

Proposition 3.12 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space. Then for all $x \in B$

i) $x^{\vee}_{\varphi} \leq x \leq x^{\wedge}_{\varphi}$

ii) $0^{\vee}_{\varphi} = 0 = 0^{\wedge}_{\varphi}$

iii) $1^{\vee}_{\varphi} = 1 = 1^{\wedge}_{\varphi}$

Proof: **(i)**Assume that $b \in A(B)$, s.t $b \leq x^{\vee}_{\varphi}$, then $b \leq x$. For the other inclusion, let $b \in A(B)$ s.t $b \leq x$. Then $\varphi(b) = \varphi(b)$ for some $b \leq x$. Thus $b \leq x^{\wedge}_{\varphi}$.
(ii) $0^{\vee}_{\varphi} = \bigvee \{a \in A(B) : a \leq 0\, , \varphi(a) \neq \varphi(b)\ \forall b \in A(B)\ s.t\, b \nleq 0\} = 0$. Also, $0^{\wedge}_{\varphi} = \bigvee \{a \in A(B) : \varphi(a) = \varphi(b)\ \ for\ some\, b \in A(B)\ s.t\, b \leq 0\} = 0$. Claim (iii) can be proved similarly.

Proposition 3.13 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space. Then for all $x, y \in B$

i) The mappings ${}^{\vee}_{\varphi} : B \longrightarrow B$ and ${}^{\wedge}_{\varphi} : B \longrightarrow B$ are order preserving.

ii) The mappings ${}^{\vee}_{\varphi} : B \longrightarrow B$ and ${}^{\wedge}_{\varphi} : B \longrightarrow B$ are mutually dual.

Proof: i) Assume that $x \leq y$ and $a \leq x^{\vee}_{\varphi}$. Let $b \in A(B)$ s.t $b \nleq y$. Since $x \leq y$, then $b \nleq x$. Since $a \leq x^{\vee}_{\varphi}$, then $\varphi(a) \neq \varphi(b)$. So $a \leq y^{\vee}_{\varphi}$ and we get $x^{\vee}_{\varphi} \leq y^{\vee}_{\varphi}$. Now let $b \leq x^{\wedge}_{\varphi}$, that is $\varphi(b) = \varphi(c)$ for some $c \in A(B)$ s.t $c \leq x$. Since $x \leq y$, then $\varphi(b) = \varphi(c)$ for some $c \in A(B)$ s.t $c \leq y$. Thus $b \leq y^{\wedge}_{\varphi}$. Consequently, $x^{\wedge}_{\varphi} \leq y^{\wedge}_{\varphi}$.
ii) We must show that $x^{\wedge}_{\varphi} = ((x')^{\vee}_{\varphi})'$ and $((x')^{\wedge}_{\varphi})' = x^{\vee}_{\varphi}$. Let $a \in A(B)$ s.t $a \leq ((x')^{\vee}_{\varphi})'$, then $a \nleq (x')^{\vee}_{\varphi}$. So, either $a \nleq x$ or $\varphi(a) = \varphi(b)$ for some $b \in A(B)$ s.t $b \leq (x')' = x$, that is $a \leq x^{\wedge}_{\varphi}$.
Conversely, Let $a \in x^{\wedge}_{\varphi}$, then $\varphi(a) = \varphi(b)$ for some $b \in A(B)$ s.t $b \leq x$. Thus $a \nleq (x')^{\vee}_{\varphi}$, that is $a \leq ((x')^{\vee}_{\varphi})'$.
Let $a \in A(B)$ s.t $a \leq ((x')^{\wedge}_{\varphi})'$, then $a \nleq (x')^{\wedge}_{\varphi}$. Hence $\varphi(a) \neq \varphi(b)$ for all $b \in A(B)$ s.t $b \leq x'$. That is $a \leq x$ because otherwise, if $a \leq x'$, then $\varphi(a) \neq \varphi(a)$, a contradiction. Thus $a \leq x$, $\varphi(a) \neq \varphi(b)$ for all $b \in A(B)$ s.t $b \leq x'$ and consequently, $a \leq x^{\vee}_{\varphi}$.
Conversely, let $a \in A(B)$ s.t $a \leq x^{\vee}_{\varphi}$, then $a \leq x$, $\varphi(a) \neq \varphi(b)$ for all $b \in A(B)$ s.t $b \leq x'$. Thus $a \nleq (x')^{\wedge}_{\varphi}$, i.e $a \leq ((x')^{\wedge}_{\varphi})'$.

Corollary 3.14 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space. Then f_A is full iff $x \leq x^\wedge$ for every $x \in B$.
Proof: $(\Rightarrow)$ Assume that f_A is full and $x \in B$. Then $x \leq x^\wedge_\varphi \leq x^\wedge$ (by Propositions 3.8 and 3.12).
$(\Leftarrow)$ Assume that $x \leq x^\wedge$ for every $x \in B$. Let $b \in A(B)$, then $b \leq b^\wedge \leq \bigvee\{f(e) : e \in A \, and f(e) \wedge b \neq 0\} = \bigvee\{f(e) : e \in A \, and \, b \leq f(e)\}$. Therefore $\exists\ e \in A$ s.t $b \leq f(e)$ and consequently, f_A is full.

For all $S \subseteq B$, we denote $S^\vee_\varphi = \{x^\vee_\varphi : x \in S\}$ and $S^\wedge_\varphi = \{x^\wedge_\varphi : x \in S\}$.

Proposition 3.15 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space, then

i) For all $S \subseteq B$, $\vee S^\wedge_\varphi = (\vee S)^\wedge_\varphi$ and $\wedge S^\wedge_\varphi \geq (\wedge S)^\wedge_\varphi$.

ii) For all $S \subseteq B$, $\wedge S^\vee_\varphi = (\wedge S)^\vee_\varphi$ and $\vee S^\vee_\varphi \leq (\vee S)^\vee_\varphi$.

iii) $(B^\wedge_\varphi, \leq)$ is a complete lattice; 0 is the least element and $1^\wedge_\varphi$ is the greatest element of $(B^\wedge_\varphi, \leq)$.

iv) $(B^\vee_\varphi, \leq)$ is a complete lattice; 0 is the least element and $1^\vee_\varphi$ is the greatest element of$(B^\vee_\varphi, \leq)$.

v) The kernal $\Theta^\vee_\varphi = \{(x, y) : x^\vee_\varphi = y^\vee_\varphi\}$ of the map ${}^\vee_\varphi : B \longrightarrow B$ is a congruence on the semi lattice $(B, \wedge)$ such that the $\Theta^\vee_\varphi$-class of any x has a least element.

vi) The kernal $\Theta^\wedge_\varphi = \{(x, y) : x^\wedge_\varphi = y^\wedge_\varphi\}$ of the map ${}^\wedge_\varphi : B \longrightarrow B$ is a congruence on the semi lattice $(B, \vee_\varphi)$ such that the $\Theta^\wedge_\varphi$-class of any x has a least element.

Proof: (i) Let $S \subseteq B$. The mapping ${}^\wedge_\varphi{:}B \longrightarrow B$ is order preserving, which implies that $\vee S^\wedge_\varphi \leq (\vee S)^\wedge_\varphi$ and $\wedge S^\wedge_\varphi \geq (\wedge S)^\wedge_\varphi$. Let $b \in A(B)$ and assume that $a \leq (\vee S)^\wedge_\varphi$. So, $\varphi(a) = \varphi(b)$ for some $b \in A(B)$ s.t $b \leq \vee S$. So, $\varphi(a) = \varphi(b)$ for some $b \in A(B)$ and $x \in S$ s.t $b \leq x$. So $\{a \in A(B) : \varphi(a) = \varphi(b) \, forsome \, b \in A(B) \, s.t \, b \leq \vee S\}$
$\subseteq \cup_{x \in S}\{a \in A(B) : \varphi(a) = \varphi(b) forsome \, b \in A(B) \, s.t \, b \leq x\}$. Then
$(\vee S)^\wedge_\varphi = \bigvee\{a \in A(B) : \varphi(a) = \varphi(b) \; for \; some \; b \in A(b) \; s.t \; b \leq \vee S\}$
$\leq \bigvee(\cup_{x \in S}\{a \in A(B) : \varphi(a) = \varphi(b) \; for \; some \; b \in A(B) \; s.t \; b \leq \vee S\})$
$= \bigvee_{x \in S}(\bigvee\{a \in A(B) : \varphi(a) = \varphi(b) \; for \; some \; b \in A(b) \; s.t \; b \leq \vee S\})$ (by Lemma 2.1)
$=\bigvee\{x^\wedge_\varphi : x \in S\} = \bigvee S^\wedge_\varphi$
(ii) Let $S \subseteq B$. The mapping ${}^\vee_\varphi{:}B \longrightarrow B$ is order preserving, which implies that $(\wedge S)^\vee_\varphi \leq \wedge S^\vee_\varphi$ and $\vee S^\vee_\varphi \leq (\vee S)^\vee_\varphi$. Let $a \in A(B)$ s.t $a \leq \wedge S^\vee_\varphi = \wedge\{x^\vee_\varphi : x \in S\}$. So, $a \leq x$ and $\varphi(a) \neq \varphi(b)$ for all $b \in A(B)$ s.t $b \not\leq x$ for every $x \in S$. Hence $\varphi(a) \neq \varphi(b)$ for all $b \in A(B)$ s.t $b \not\leq \wedge S$. In fact if $b \not\leq \wedge S$, then $\exists\ x \in S$ s.t $b \not\leq x$. So, $\varphi(a) \neq \varphi(b)$. Therefore

$b \leq (\wedge S)^{\vee}_{\varphi}$. Consequently, $\wedge S^{\vee}_{\varphi} \leq (\wedge S)^{\vee}_{\varphi}$. Assertions (iii) and (iv) follow easily from (i), (ii) and Proposition 3.12(i). The proof of (v) and (vi) follow by (i) and (ii).

The inequality in and in Proposition 3.15 may be proper. This can be seen in the following example.

Example 3.16 Let $B = \{0, a, b, c, d, e, f, 1\}$ and let the order $\leq$ be defined as in figure 1.
Let $A = \{e_1, e_2, e_3, \}$ and f_A be a soft set over B defined as follows:
$f(e_1) = 0$, $f(e_2) = c$, and $f(e_3) = d$. Then the map φ of MSR-approximation space $(A(B), \varphi)$ will be $\varphi(a) = \{e_3\}$, $\varphi(b) = \{e_3\}$, and $\varphi(c) = \{e_2\}$.
If we take $x = d$ and $y = f$. Then $x \vee y = 1$ and $(x \vee y)^{\vee}_{\varphi} = 1$. Also, $x^{\vee}_{\varphi} = c$, $y^{\vee}_{\varphi} = c$ and $x^{\vee}_{\varphi} \vee y^{\vee}_{\varphi} = c$. Thus $(x \vee y)^{\vee}_{\varphi} >_{\neq} x^{\vee}_{\varphi} \vee y^{\vee}_{\varphi}$.
Now $x^{\wedge}_{\varphi} = a \vee b = d$, $y^{\wedge}_{\varphi} = a \vee b = d$. So $x^{\wedge}_{\varphi} \wedge y^{\wedge}_{\varphi} = d$. On the other hand $x \wedge y = e \wedge f = c$ and $(x \wedge y)^{\wedge}_{\varphi} = 0$. Thus $x^{\wedge}_{\varphi} \wedge y^{\wedge}_{\varphi} >_{\neq} (x \wedge y)^{\wedge}_{\varphi}$.

Proposition 3.17 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space. Then $(B^{\vee}_{\varphi}, \geq) \cong (B^{\wedge}_{\varphi}, \leq)$

Proof: We show that $x^{\wedge}_{\varphi} \longrightarrow (x^{'})^{\vee}_{\varphi}$ is the required dual order isomorphism. It is obvious that $x^{\wedge}_{\varphi} \longrightarrow (x^{'})^{\vee}_{\varphi}$ is onto $(B^{\vee}_{\varphi}, \geq)$. We show that $x^{\wedge}_{\varphi} \longrightarrow (x^{'})^{\vee}_{\varphi}$ is order embedding. Suppose that $x^{\wedge}_{\varphi} \leq y^{\wedge}_{\varphi}$. Then for all $a \in A(B)$, $a \leq x^{\wedge}_{\varphi}$ implies $a \leq y^{\wedge}_{\varphi}$. So, for all $a \in A(B)$ such that $\varphi(a) = \varphi(c)$ for some $c \in A(B)$ s.t $c \leq x$ implies $\varphi(a) = \varphi(b)$ for some $b \in A(B)$ s.t $b \leq y$. Suppose that $(y^{'})^{\vee}_{\varphi} \not\leq (x^{'})^{\vee}_{\varphi}$. So there exists $a \in A(B)$ such that $a \leq (y^{'})^{\vee}_{\varphi}$ and $a \not\leq (x^{'})^{\vee}_{\varphi}$. Since $a \leq (y^{'})^{\vee}_{\varphi}$, then $a \leq y'$ and $\varphi(a) \neq \varphi(b)$ for all $b \in A(B)$ s.t $b \leq (y^{'})^{'} = y$. Also $a \not\leq (x^{'})^{\vee}_{\varphi}$ implies either $a \not\leq x'$ or $\varphi(a) = \varphi(c)$ for some $c \in A(B)$ s.t $c \leq (x^{'})^{'} = x$, a contradiction. Hence $(y^{''})^{\vee}_{\varphi} \leq (x^{'})^{\vee}_{\varphi}$. On the other hand, assume that $(y^{'})^{\vee}_{\varphi} \leq (x^{'})^{\vee}_{\varphi}$ and $x^{\wedge}_{\varphi} \not\leq y^{\wedge}_{\varphi}$. So there exists $a \in A(B)$ such that $a \leq x^{\wedge}_{\varphi}$ and $a \not\leq y^{\wedge}_{\varphi}$. So $\varphi(a) = \varphi(b)$ for some $b \in A(B)$ s.t $b \leq x$. But $\varphi(a) \neq \varphi(c)$ for all $c \in A(B)$ s.t $c \leq y$, a contradiction.

Proposition 3.18 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space. Then for all $x \in B$

i) $(x^{\vee}_{\varphi})^{\vee}_{\varphi} = x^{\vee}_{\varphi}$;

ii) $(x^{\wedge}_{\varphi})^{\wedge}_{\varphi} = x^{\wedge}_{\varphi}$.

Proof: i) $(x^{\vee}_{\varphi})^{\vee}_{\varphi} = \bigvee \{a \in A(B) : a \leq x^{\vee}_{\varphi}\,, \varphi(a) \neq \varphi(b)\, \forall\, b \in A(B)\ s.t\, b \not\leq x^{\vee}_{\varphi}\}$. By Proposition 3.12 $x^{\vee}_{\varphi} \leq x$ which gives $x^{'} \leq (x^{\vee}_{\varphi})^{'}$. So if $b \leq x^{'}$, then $b \leq (x^{\vee}_{\varphi})^{'}$. Therefore $(x^{\vee}_{\varphi})^{\vee}_{\varphi} = \bigvee \{a \in A(B) : a \leq x^{\vee}_{\varphi}\,, \varphi(a) \neq \varphi(b)\, \forall\, b \in A(B)\ s.t\, b \not\leq x\} = x^{\vee}_{\varphi}$.
ii)By Proposition 3.12 $x^{\wedge}_{\varphi} \leq (x^{\wedge}_{\varphi})^{\wedge}_{\varphi}$. For the reverse inclusion, let $a \leq (x^{\wedge}_{\varphi})^{\wedge}_{\varphi}$, then $\varphi(a) = \varphi(b)$ for some $b \in A(B)$ s.t $b \leq x^{\wedge}_{\varphi}$, that is $\varphi(b) = \varphi(c)$ for some $c \in A(B)$ s.t $c \leq x$.

This implies that $\varphi(a)=\varphi(c)$ for some $c\in A(B)$ s.t $c\leq x$. Hence $a\leq x_{\varphi}^{\wedge}$ and therefore $(x_{\varphi}^{\wedge})_{\varphi}^{\wedge}\leq x_{\varphi}^{\wedge}$. This implies that $(x_{\varphi}^{\wedge})_{\varphi}^{\wedge}=x_{\varphi}^{\wedge}$.

Proposition 3.19 Let $\mathbf{B}=(B,\leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B),\varphi)$ be a MSR-approximation space. Then for all $x\in B$

i) $(x_{\varphi}^{\vee})_{\varphi}^{\wedge}=x_{\varphi}^{\vee}$

ii) $(x_{\varphi}^{\wedge})_{\varphi}^{\vee}=x_{\varphi}^{\wedge}$

Proof: i)$x_{\varphi}^{\vee}\leq(x_{\varphi}^{\vee})_{\varphi}^{\wedge}$ by Proposition 3.12. For the converse assume that $a\leq(x_{\varphi}^{\vee})_{\varphi}^{\wedge}$. So $\varphi(a)=\varphi(b)$ for some $b\in A(B)$ s.t $b\leq x_{\varphi}^{\vee}$. Hence $\varphi(b)\neq\varphi(c)$ for all $c\in A(B)$ s.t $c\leq x^c$. Since $\varphi(a)=\varphi(b)$, then $\varphi(a)\neq\varphi(c)$ for all $c\in A(B)$ s.t $c\leq x^c$. Since $\varphi(a)=\varphi(a)$, then $a\not\leq x^c$, i.e $a\leq x$. So $a\leq x_{\varphi}^{\vee}$. Hence $(x_{\varphi}^{\vee})_{\varphi}^{\wedge}\leq x_{\varphi}^{\vee}$ and we conclude that $(x_{\varphi}^{\vee})_{\varphi}^{\wedge}=x_{\varphi}^{\vee}$.
ii)By Proposition 3.12 $(x_{\varphi}^{\wedge})_{\varphi}^{\vee}\leq x_{\varphi}^{\wedge}$. Conversely, if $a\not\leq(x_{\varphi}^{\wedge})_{\varphi}^{\vee}$, then either $a\not\leq x_{\varphi}^{\wedge}$ or $\varphi(a)=\varphi(b)$ for some $b\in A(B)$ s.t $b\leq(x_{\varphi}^{\wedge})^c$. If $a\not\leq x_{\varphi}^{\wedge}$, then we get our required result. In later case, $\varphi(a)=\varphi(b)$ for some $b\in A(B)$ s.t $b\leq(x_{\varphi}^{\wedge})^c$. So that $b\not\leq(x_{\varphi}^{\wedge})$ i.e $\varphi(b)\neq\varphi(z)$ for all $z\in A(B)$ s.t $z\leq x$. Therefore $\varphi(a)\neq\varphi(z)$ for all $z\in A(B)$ s.t $z\leq x$. Also, $a\leq x^c$ because if $a\not\leq x^c$, then $a\leq x$. So $\varphi(a)=\varphi(b)\neq\varphi(a)$ a contradiction. Hence $a\leq(x^c)_{\varphi}^{\vee}$. That is $a\not\leq((x^c)_{\varphi}^{\vee})^c=x_{\varphi}^{\wedge}$ by Proposition 3.13.

Corollary 3.20 Let $\mathbf{B}=(B,\leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B),\varphi)$ be a MSR-approximation space. Then the pair $({}_{\varphi}^{\vee},{}_{\varphi}^{\wedge})$ is a dual Galois connection on B
Proof: By Propositions and $(x_{\varphi}^{\vee})_{\varphi}^{\wedge}=x_{\varphi}^{\vee}\leq x\leq x_{\varphi}^{\wedge}=(x_{\varphi}^{\wedge})_{\varphi}^{\vee}$.

Proposition 3.21 Let $\mathbf{B}=(B,\leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B),\varphi)$ be a MSR-approximation space, then

i) The map ${}_{\varphi}^{\wedge}:B\longrightarrow B$ is a closure operator.

ii) The map ${}_{\varphi}^{\vee}:B\longrightarrow B$ is an interior operator.

iii) $(B_{\varphi}^{\vee},\leq)$ and $(B_{\varphi}^{\wedge},\leq)$ are sublattices of $(B,\leq)$.

Proof: i) The mapping ${}_{\varphi}^{\wedge}:B\longrightarrow B$ is extensive, and it is order-preserving by Proposition 3.12. By Proposition 3.18, $(x_{\varphi}^{\wedge})_{\varphi}^{\wedge}=x_{\varphi}^{\wedge}$. Claim (ii) follows from Lemma 2.3 and Proposition 3.17.
iii) Suppose that $x_{\varphi}^{\vee},\,y_{\varphi}^{\vee}\in B_{\varphi}^{\vee}$. Then obviously, $x_{\varphi}^{\vee}\wedge y_{\varphi}^{\vee}=(x\wedge y)_{\varphi}^{\vee}$ which implies that $x_{\varphi}^{\vee}\wedge y_{\varphi}^{\vee}\in B_{\varphi}^{\vee}$. Next we show that $x_{\varphi}^{\vee}\vee y_{\varphi}^{\vee}=(x_{\varphi}^{\vee}\vee y_{\varphi}^{\vee})_{\varphi}^{\vee}$. It is clear that $x_{\varphi}^{\vee}\leq x_{\varphi}^{\vee}\vee y_{\varphi}^{\vee}$ and $x_{\varphi}^{\vee}=(x_{\varphi}^{\vee})_{\varphi}^{\vee}\leq(x_{\varphi}^{\vee}\vee y_{\varphi}^{\vee})_{\varphi}^{\vee}$. Similarly, we can show that $y_{\varphi}^{\vee}\leq(x_{\varphi}^{\vee}\vee y_{\varphi}^{\vee})_{\varphi}^{\vee}$. Thus, $(x_{\varphi}^{\vee}\vee y_{\varphi}^{\vee})_{\varphi}^{\vee}$ is an upper bound of $x_{\varphi}^{\vee}$ and $x_{\varphi}^{\vee}$. Let $z\in B$ be an upper bound of $x_{\varphi}^{\vee}$ and $x_{\varphi}^{\vee}$. Then $x_{\varphi}^{\vee}\leq z$ and $y_{\varphi}^{\vee}\leq z$, which implies $x_{\varphi}^{\vee}\vee y_{\varphi}^{\vee}\leq z$. So, $(x_{\varphi}^{\vee}\vee y_{\varphi}^{\vee})_{\varphi}^{\vee}\leq x_{\varphi}^{\vee}\vee y_{\varphi}^{\vee}\leq z$. Thus, $x_{\varphi}^{\vee}\vee y_{\varphi}^{\vee}=(x_{\varphi}^{\vee}\vee y_{\varphi}^{\vee})_{\varphi}^{\vee}$ and $x_{\varphi}^{\vee}\vee y_{\varphi}^{\vee}\in B_{\varphi}^{\vee}$. The other part can be proved analogously.

Since every sublattice of a distributive lattice is distributive (see[5], for example). Therefore, we can write the following corollary.

Corollary 3.22 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space, then $(B_\varphi^\vee, \leq)$ and $(B_\varphi^\wedge, \leq)$ are distributive.

Remark 3.22 1) It is mentioned that in order to prove that for all $S \subseteq B$, $\wedge S^\vee = (\wedge S)^\vee$ in [45] we employed a strong condition on soft set f_A over a complete atomic Boolean lattice to be keeping infimum. However in proving $\wedge S_\varphi^\vee = (\wedge S)_\varphi^\vee$ no such condition is required.

2) It is clear that MSR-element over a complete atomic Boolean lattice satisfies all the basic properties similar to rough element introduced by Jarvinen [45]. Thus, MSR-element over a complete atomic Boolean lattice provides a good combination of roughness and parametrization.

3. Relation between MSR sets and Rough sets on a Complete atomic Boolean Lattice

In the following, we introduce the notion of Boolean lattice information system and we show that every soft sets on a complete atomic Boolean lattice induces a Boolean lattice information system and vice versa.

Definition 4.1 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and A be a finite set of attributes. The pair $(A(B), A, V, g)$ is called lattice information system, if g is an information function from $A(B) \times A$ to $V = \bigcup_{e \in A} V_e$ where $V_e = \{g(b, e) : b \in A(B), e \in A\}$ is the values of the attribute set e.

Definition 4.2 A lattice information system $(A(B), A, V, g)$ is called Boolean lattice information system if $V = \{0, 1\}$.

Definition 4.3 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $S = f_A$ be a soft set over B. Then f_A induces a Boolean lattice information system $I_s = (A(B), A, V, g_s)$, where $g_s : A(B) \times A \longrightarrow V = \{0, 1\}$, For any $b \in A(B)$ and $e \in A$,

$$g_s(b, e) = \begin{cases} 1 & \text{if } b \leq f(e), \\ 0 & \text{if } b \not\leq f(e) \end{cases}$$

Definition 4.4 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and $I = (A(B), A, V, g)$ be a lattice information system. Then $S^I = f_A^I$ is called a soft set over B induced by I, where $f_A^I : A \longrightarrow B$ and for $e \in A$, $f^I(e) = \vee\{b \in A(B) : g(b, e) = 1\}$.

Proposition 4.5 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and $= f_A$ be a soft set over B. Let $I_s = (A(B), A, V, g_s)$ be a Boolean lattice information system induced by f_A and $S^{I_s} = f_A^{I_s}$ be a soft set over B induced by I_s. Then $f_A^{I_s} = f_A^I$.

Proof: By Definition 4.4, for any $e \in A$, $f_A^{I_s}(e) = \vee\{b \in A(B) : g_s(b, e) = 1\}$.
By Definition 4.3, for any $b \in A(B)$ and $e \in A$,

$$g_s(b, e) = \begin{cases} 1 & \text{if } b \leq f(e), \\ 0 & \text{if } b \not\leq f(e) \end{cases}$$

This implies that $g_s(b, e) = 1 \Leftrightarrow b \leq f(e)$. So, for any $b \in A(B)$ $e \in A$, $f(e) = f^{I_s}(e)$.

Proposition 4.6 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and $I = (A(B), A, V, g)$ be a lattice information system. Let $S^I = f_A^I$ be a soft set over B induced by I and $I_{s^I} = (A(B), A, V, g_{s^I})$ be a Boolean lattice information system induced by S^I. Then $I = I_{s^I}$.

Proof: By Definition 4.3, for any $b \in A(B)$ and $e \in A$,

$$g_{s^I}(b, e) = \begin{cases} 1 & \text{if } b \leq f^I(e), \\ 0 & \text{if } b \not\leq f^I(e) \end{cases}$$

By Definition 4.4, for any $e \in A$, $f^I(e) = \vee\{b \in A(B) : g(b, e) = 1\}$. Since $I = (A(B), A, V, g)$ be a Boolean lattice information system, then $g(b, e) = 0$ if $b \not\leq f^I(e)$. This implies that

$$g(b, e) = \begin{cases} 1 & \text{if } b \leq f^I(e), \\ 0 & \text{if } b \not\leq f^I(e) \end{cases}$$

So for any $b \in A(B)$ and $e \in A$, $g_{s^I}(b, e) = g(b, e)$. Hence $g_{s^I} = g$ and Consequently, $I = I_{s^I}$.

Definition 4.7 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $S = f_A$ be a soft set over B. Let $I_s = (A(B), A, V, g_s)$ be a Boolean lattice information system induced by $S = f_A$. Then I_s induces a mapping $\psi_s : A(B) \longrightarrow B$ as follows; for every $a, b \in A(B)$

$$a \leq \psi_s(b) \;\Leftrightarrow\; \forall\, e \in A \; g_s(a, e) = g_s(b, e)$$

Definition 4.8[45] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Define a mapping $\psi_f : A(B) \rightarrow B$ by

$$c \leq \psi_f(b) \Leftrightarrow \exists e \in A,\; s.t\; c \leq f(e) \; and \; b \leq f(e)$$

for every $c, b \in A(B)$. Then ψ_f is called the mapping induced by f_A on B.

Definition 4.9[45] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_A be a soft set over B. Let $\psi_f : A(B) \to B$ be the mapping induced by f_A on B. We define a pair of soft approximation operators operators ${}^{\nabla_f}, {}^{\triangle_f} : B \to B$ as follows

$x^{\nabla_f} = \bigvee \{b \in A(B) : \psi_f(b) \leq x\}$, and

$x^{\triangle_f} = \bigvee \{b \in A(B) : \psi_f(b) \wedge x \neq 0\}$.

The elements x^{∇_f} and $x^{\triangle_f}$ are called the **soft lower** and the **soft upper** approximations of x with respect to the mapping ψ_f induced by f_A respectively. Two elements x and y are called equivalent if they have the same soft upper and lower approximations with respect to the mapping ψ_f induced by f_A on B. The resulting equivalence classes are called soft rough sets with respect to the mapping ψ_f induced by f_A on B.

Proposition 4.10 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and $S = f_A$ be a partition soft set over B. Let $I_s = (A(B), A, V, g_s)$ be a Boolean lattice information system induced by f_A. Then

$$a \leq \psi_s(b) \Leftrightarrow a \leq \psi_f(b) \; \forall \, a, b \in A(B)$$

Proof: $(\Rightarrow)$ Let $a, b \in A(B)$ s.t $a \leq \psi_s(b)$. Then $\forall \, e \in A \; g_s(a, e) = g_s(b, e)$. Since $a \leq 1$ and f_A be a partition soft set, then $\exists \; e \in A$ s.t $a \leq f(e)$. So $g(b, e) = g(a, e) = 1$ and therefore $b \leq f(e)$. Consequently, $a \leq \psi_f(b)$.
$(\Leftarrow)$Let $a, b \in A(B)$ s.t $a \leq \psi_f(b)$. Then $\exists \; e_1 \in A$ s.t $a \leq f(e_1)$ and $b \leq f(e_1)$. So $g_s(a, e_1) = g_s(b, e_1)$. For every $e_2 \in A - e_1$, if $f(e_1) = f(e_2)$, then $a \leq f(e_2)$ and $b \leq f(e_2)$ and thus $g_s(a, e_2) = 1 = g_s(b, e_2)$. If $f(e_1) \neq f(e_2)$, then $f(e_1) \wedge f(e_2) = 0$. Since $a \leq f(e_1)$ and $b \leq f(e_1)$, then $a \not\leq f(e_2)$ and $b \not\leq f(e_2)$ and therefore $g_s(a, e_2) = 0 = g_s(b, e_2)$. So, $g_s(a, e) = g_s(b, e) \; \forall e \in A$ and consequently $a \leq \psi_s(b)$.

Definition 4.11 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $S = f_A$ be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space and $I_s = (A(B), A, V, g_s)$ be a Boolean lattice information system induced by $S = f_A$. Then

i) $\forall a, b \in A(B) \; a \leq \psi_s(b) \Leftrightarrow \varphi(a) = \varphi(b)$.

ii) $\forall x \in B$, $x_\varphi^\vee = x_s^\nabla$, where $x_s^\nabla = \bigvee \{b \in A(B) : \psi_s(b) \leq x\}$.

iii) $\forall x \in B$, $x_\varphi^\wedge = x_s^\triangle$, where $x_s^\triangle = \bigvee \{b \in A(B) : \psi_s(b) \wedge x \neq 0\}$

Proof:i)$(\Rightarrow)$ Let $a, b \in A(B)$ s.t $a \leq \psi_s(b)$. So $g_s(a, e) = g_s(b, e) \; e \in A$. Let $e \in \varphi(a)$, so $a \leq f(e)$ and hence $g_s(a, e) = 1$. Therefore $g_s(b, e) = 1$ and thus $b \leq f(e)$. Consequently, $e \in \varphi(b)$. So $\varphi(a) \subseteq \varphi(b)$, similarly we can show that $\varphi(b) \subseteq \varphi(a)$.

Table 1: Lattice information system of the soft set F_A

	e_1	e_2	e_3
a	0	1	0
b	1	0	1
c	0	1	0

($\Leftarrow$) Let $a, b \in A(B)$ s.t $\varphi(a) = \varphi(b)$. Let $e \in A$, if $g_s(a, e) = 1$, then $a \leq f(e)$ and hence $e \in \varphi(a) = \varphi(b)$. Also, $b \leq f(e)$ and so $g_s(b, e) = 1$. If $g_s(a, e) = 0$, then $a \nleq f(e)$ and hence $e \notin \varphi(a) = \varphi(b)$. Therefore $b \nleq f(e)$ and thus $g_s(b, e) = 0$. So $g_s(a, e) = g_s(b, e)$ $\forall$ $e \in A$ and consequently, $a \leq \psi_s(b)$.
ii) Let $x \in B$ and $b \in A(B)$ s.t $b \leq x_\varphi^\vee$. We show that $\psi_s(b) \leq x$. Let $a \in A(B)$ s.t $a \leq \psi_s(b)$, then $\varphi(a) = \varphi(b)$ by i). But $b \leq x_\varphi^\vee$ implies that $\varphi(b) \neq \varphi(c)$ $\forall$ $c \in A(B)$ s.t $c \nleq x$. Since $\varphi(a) = \varphi(b)$, then $a \leq x$ and therefore $\varphi_s(b) \leq x$. This implies that $x_\varphi^\vee \leq x_s^\nabla$. Conversely, Let $b \in A(B)$ s.t $b \leq x_s^\nabla$. So $\psi_s(b) \leq x$. Let $a \in A(B)$ s.t $a \nleq x$, thus $a \nleq \psi_s(b)$. Therefore $\varphi(a) \neq \varphi(b)$ by i) and so $b \leq x_\varphi^\vee$. Consequently, $x_s^\nabla \leq x_\varphi^\vee$.
iii) Let $x \in B$ and $b \in A(B)$ s.t $b \leq x_\varphi^\wedge$, then $\exists$ $a \in A(B)$ s.t $a \leq x$ and $\varphi(a) = \varphi(b)$. So $a \leq \psi_s(b)$ and therefore $\psi_s(b) \wedge x \neq 0$. Consequently, $b \leq x_s^\triangle$.
Conversely, let $b \in A(B)$ s.t $b \leq x_s^\triangle$, then $\psi_s(b) \wedge x \neq 0$. Thus $\exists$ $a \in A(B)$ s.t $a \leq x$ and $a \leq \psi_s(b)$. Thus $\varphi(a) = \varphi(b)$ by i) and therefore $b \leq x_\varphi^\wedge$.

3. Modified rough approximations of soft sets on a complete atomic boolean lattice

In this section, we introduce the concept of Boolean lattice information system with respect to another Boolean lattice information system. We study upper and lower MSR-approximations of soft set on a complete atomic Boolean lattice with respect to another soft set.

Definition 5.1 Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let f_{A_1} be a soft set over B. Let $(A(B), \varphi)$ be a MSR-approximation space where $\varphi : A(B) \longrightarrow P(A_1)$ is defined as $\varphi(b) = \{a \in A : b \leq f(a)\}$. Let g_{A_2} be another soft set over B. For any $e \in A_2$, lower and upper MSR of g_{A_2} over B are denoted by $(g_{A_2})_\varphi^\vee$ and $(g_{A_2})_\varphi^\wedge$ defined as

$$g(e)_\varphi^\vee = \bigvee \{a \in A(B) : a \leq g(e)\ , \varphi(a) \neq \varphi(b)\ \forall\, b \in A(B)\ s.t\, b \nleq g(e)\}\ \forall e \in A_2,$$

$$g(e)_\varphi^\wedge = \bigvee \{a \in A(B) : \varphi(a) = \varphi(b)\ for\ some\ b \in A(b)\ s.t\, b \leq g(e)\}\ \forall e \in A_2.$$

Example 5.2 Let $B = \{0, a, b, c, d, e, f, 1\}$ representing 8 patients, where 0 denotes patients who drink mineral water only, a denotes patients who drink coffee, b denotes

patients who drink cola, c denotes patients who drink tea, d denotes patients who drink caffeine drinks e denotes patients who drink antioxidant drinks and f denotes patients who drink cold drinks. So, the order $\leq$ can be defined as in figure 1.
Let $A_1 = \{e_1, e_2, e_3, \}$, where e_1 denotes temperature, e_2 denotes headache and e_3 denotes stomach problem and let f_{A_1} be a soft set over B representing the diagnosis of doctor M, defined as follows:
$f(e_1) = b$, $f(e_2) = e$ and $f(e_3) = b$. Then the Boolean lattice information system of f_{A_1} can be given by table 1, where 1 and 0 denote yes and no respectively.
Then the map φ of MSR-approximation space $(A(B), \varphi)$ will be $\varphi(a) = \{e_2\}$, $\varphi(b) = \{e_1, e_3\}$, and $\varphi(c) = \{e_2\}$.
Let $A_2 == \{e_1, e_2, e_3, e_4\}$ and g_{A_2} be another soft set over B representing the diagnosis of doctor N , where $A_2 = \{e_1, e_2, e_3, e_4\}$ and e_4 represents cough, defined as follows:
$g(e_1) = d$, $g(e_2) = b$, $g(e_3) = c$ and $g(e_4) = e$
lower MSR of g_{A_2} over B are
$g(e_1)^{\vee}_{\varphi} = b$, $g(e_2)^{\vee}_{\varphi} = b$, $g(e_3)^{\vee}_{\varphi} = 0$ and $g(e_4)^{\vee}_{\varphi} = a$.
upper MSR of g_{A_2} over B are
$g(e_1)^{\wedge}_{\varphi} = a \vee b \vee c = 1$, $g(e_2)^{\wedge}_{\varphi} = b$, $g(e_3)^{\wedge}_{\varphi} = a \vee c = e$ and $g(e_4)^{\wedge}_{\varphi} = e$.

3. Conclusion

Lattice is a very important structure in mathematics. We introduced the concept of soft sets on a complete atomic Boolean lattice B. We combine soft set and rough set by introducing the concept of soft rough set on B. Some shortcoming became the part of soft rough sets on B. In this paper, we introduced the concept of Modified soft rough sets(MSR) on a complete atomic Boolean lattice. Some important properties of MSR on B have been discussed. Similar results which require some strong conditions for their proof in soft rough sets on B can be proved in MSR sets without these conditions.

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