On multi-set function and order relation

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Abstract

In this paper, we study the cartesian product of two multi-sets , multi-set function and some of its properties . We defined the image and the inverse image of multi- sets and study of some properties .We study the multi-order relation ,quasi-multi-increasing (decreasing) function,quasi-multi-increasing (decreasing) sets ,and the multi-topological ordered spaces and some of m - separation axioms of multi-topological spaces.Finally ,we study a different types of generalized rough msets and illustrate the relation between them .

keywords: Multi-set, m-cartesian product, Multi-set relation ,m-quasi-conicident , ,Multi-set function ,multi-topological space , ordered relation ,m-approximation space .

1. Introduction

In the classical set theory, a set is a well-defined collection of distinct objects .If repeated occurrences of any object is allowed in a set ,then a mathematical structure that is known as multiset (mset for short). The same idea are studied by Yager in [15] and Jena in [9], but under the name of bags and lists. Thus, a multiset differs from a set in the sense that each element has a multiplicity-a natural number not necessarily one-that indicates how many times it is the member of the multiset was studied by ([1], [5], [6], [7], [8]). One of the most natural and simplest examples is the multiset of prime factors of a positive integer "n" . The number 640 has the factorization $640 = 2^3 4^2 5^1$ which gives the multiset $\{2, 2, 2, 4, 4, 5\}$.

The theory of rough sets proposed by (Pawlak 1982 ,1991) is an extension of a set theory for the study of intelligent systems characterized by insufficient and incomplete information . The lower and the upper approximation operators are constricted by using a binary relation on a set . Using the concepts of the lower and upper approximations from the rough set theory .
In any information system ,Some situations may occur where the respective counts objects in the universe of discourse are not single .In such situations we replace its universe of discourse by multisets called rough multisets . The relation on a multisets is a mathematical model to which many real life data can be connected .In fact the multi-relations are using the construction of multi-topological spaces .

We first introduce and study in section some properties and concepts of msets and mset relation .Moreover we study the properties of cartesian product in case of msets and we define image ,inverse image of mset function in sections 3 and4 .In section 5 ,we study multi-order relation and some of its properties .In addition we study multi-order topological spaces and some of its separation axioms .Finally ,we define three generalized rough msets and give an example to illustrate the relations between different types of generalized definitions of rough multiset approximations .

2. Preliminaries and Basic definitions

In this section the basic definitions and notations of multisets and the different types of collections of multisets and the basic definitions of functions in multiset context are given ([1],[5], [6], [7],[8]).

Definition 2.1:[6] A mset M drawn from the set X is represented by a function count M or C_M defined as $C_M : X \to N$ where N represents the set of non negative integers.
Here $C_M(x)$ is the number of occurrences of the element x in the mset M. Let$X = \{x_1, x_2, ..., x_n\}$, the mset M drawn from the set X is represented by $M = \{m_1/x_1, m_2/x_2, ..., m_n/x_n\}$ where m_i is the number of occurrences of the element x_i, $i = 1, 2, ..., n$ in the mset M.

Example 2.2: Let $X = \{a, b, c, d, e\}$ be any set. Then $M = \{2/a, 3/b, 5/d, 1/e\}$ is an mset drawn from X. Clearly, a set is a special case of an mset.

Definition 2.3:[6,7] Let M and N be two msets drawn from a set X. Then the following are defined:

1. $M = N$ if $C_M(x) = C_N(x),\ \forall x \in X$.
2. $M \subseteq N$ if $C_M(x) \leq C_N(x),\ \forall x \in X$.
3. $P = M \bigcup N$ if $C_P(x) = Max\{C_M(x), C_N(x)\},\ \forall x \in X$.
4. $P = M \bigcap N$ if $C_P(x) = Min\{C_M(x), C_N(x)\},\ \forall x \in X$.
5. $P = M \oplus N$ if $C_P(x) = C_M(x) + C_N(x),\ \forall x \in X$.
6. $P = M \ominus N$ if $C_P(x) = Max\{C_M(x) - C_N(x), 0\},\ \forall x \in X$.

Where $\oplus$ and $\ominus$ represent m-set addition and m-set subtraction, respectively.

Definition 2.4:[7,8] Let M be any m-set drawn from a set X. The support set of M denoted by M^* is a subset of X and defined as $M^* = \{x \in X : C_M(x) > 0\}$ i.e. M^* is an ordinary set. M^* is also called root set.

Definition 2.5:[7,8] An m-set M is said to be an empty m-set if $C_M(x) = 0,\ \forall x \in X$.

Definition 2.6:[7,8] A domain X, is defined as a set of elements from which m-sets are constructed. The m-set space $[X]^w$ is the set of all m-sets whose elements are in X such that no element in the m-set occurs more

than w times.
The set $[X]^\infty$ is the set of all m-sets over a domain X such that there is no limit on the number of the occurrences of an element in an m-set.
If $X = \{x_1, x_2, ..., x_k\}$ then $[X]^w = \{m_1/x_1, m_2/x_2, ..., m_k/x_k\}$: for $i = 1, 2, ..., k\,; m_i \in \{0, 1, 2, ..., w\}\}$.

Definition 2.7:[7,8] Let X be a support set and $[X]^w$ be the m-set space defined over X. Then for any m-set $M \in [X]^w$, the complement M^c of M in $[X]^w$ is an element of $[X]^w$ such that
$C_{M^c}(x) = w - C_M(x),\ \forall x \in X$.

Notation 2.8:[7,8] Let M be an m-set from X with x appearing n times in M. It is denoted by $x \in^n M$. $M = \{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$ where M is an m-set with x_1 appearing k_1 times, x_2 appearing k_2 times and so on. $[M]_x$ denotes that the element x belongs to the m-set M and $\mid [M]_x \mid$ denotes the cardinality of an element x in M.

Definition 2.9:[6,8] A submset N of M is said to be:

1. Whole m-subset of M if $C_N(x) = C_M(x)$, for every $x \in N$.
2. Partial whole m-subset of M if $C_N(x) = C_M(x)$, for some $x \in N$.

3. Full submset of M if $M^* = N^*$ and $C_N(x) \leq C_M(x)$, for every $x \in N$.

Definition 2.10:[6,7,8] (Power Mset) Let $M \in [X]^w$ be an mset. The power mset of M denoted by P(M) is the set of all the submsets of M. i.e. $N \in P(M)$ if and only if $N \subseteq M$. If $N = \emptyset$, then $N \in^1 P(M)$, and

if $N \neq \emptyset$, then $N \in^k P(M)$ where $k = \prod_z \binom{|[M]_z|}{|[N]_z|}$, the product $\prod_z$ is taken over by distinct of z of the mset N and $|\, [M]_z \,| = m$ iff $z \in^m M$, $|\, [N]_z \,| = n$ iff $z \in^n N$, then

$\binom{|[M]_z|}{|[N]_z|} = \binom{m}{n} = \frac{m!}{n!(m-n)!}$.

The power set of an mset is the support set of the power mset and is denoted by $P^*(M)$.

Definition 2.11:[7,8] The maximum mset is defined as Z where $C_Z(x) = Max\{C_M(x) : x \in^k M, M \in [X]^w$ and $k \leq m\}$.

Definition 2.12:[6,7] Let $[X]^w$ be an mset space, $\{M_1, M_2, ...\}$ be a collection of msets drawn from $[X]^w$, then the following operations are possible under an arbitrary collection of msets.

1. The union $\cup_{i\in I} M_i = \{C_{\cup M_i}(x)/x : C_{\cup M_i}(x) = Max\{C_{M_i}(x) : x \in X\}\}$.
2. The intersection $\cap_{i\in I} M_i = \{C_{\cap M_i}(x)/x : C_{\cap M_i}(x) = Min\{C_{M_i}(x) : x \in X\}\}$.
3. The m-set addition $\oplus_{i\in I} M_i = \{C_{\oplus M_i}(x)/x : C_{\oplus M_i}(x) = \sum_{i\in I} C_{M_i}(x),\ x \in X\}$.
4. The m-set complement $M^c = Z \ominus M = \{C_{M^c}(x)/x : C_{M^c}(x) = C_Z(x) - C_M(x),\ x \in X\}$.

Definition 2.13:[8,9,10] Let M_1 and M_2 be two m-sets drawn from a set X, then the Cartesian product of M_1 and M_2 is defined as $M_1 \times M_2 = \{(m/x, n/y)/mn : x \in^m M_1,\ y \in^n M_2\}$.

Definition 2.14:[7,8,9] A sub-mset R of $M \times M$ is said to be an m-set relation on M if every member $(m/x, n/y)$ of R has a count, product of $C_1(x, y)$ and $C_2(x, y)$. We denote m/x related to n/y by $m/x\, R\, n/y$.

The Domain and Range of the m-set relation R on M is defined as follows:
Dom $R = \{x \in^k M : \exists\, y \in^r M$ such that $k/x\, R\, r/y\}$ where $C_{DomR}(x) = sup\{C_1(x, y) : x \in^k M\}$.
Ran $R = \{y \in^r M : \exists\, x \in^k M$ such that $k/x\, R\, r/y\}$ where $C_{RanR}(y) = sup\{C_2(x, y) : y \in^r M\}$.

Definition 2.15:[8,9]let R be a m-set relation on M .The family $R^{-1} = \{(n/y, m/x) : m/xRn/y\}$is said to be the inverse m-set relation of R .

Definition 2.16:[8,9]m-set relation R on M is said to be :

1. reflexive if it satisfies $m/xRm/x\ \forall\ m/x \in M$.
2. symmetric if it satisfies $m/xRn/y \Rightarrow n/yRm/x$.
3. transitive if it satisfies $m/xRn/y, n/yRk/z \Rightarrow m/xRk/z$.

A m-set relation R on M is said to be equivalence relation if it is reflexive, symmetric and transitive .

Definition 2.17:[7,8] An m-set relation

f is called an m-set function if for every element m/x in domain f, there is exactly one n/y in Ran f such that $(m/x, n/y)$ is in f with the pair occurring as the product of $C_1(x, y)$ and $C_2(x, y)$.

Definition 2.18:[7,8] Let $M \in [X]^w$ and $\tau \subseteq P^*(M)$. Then τ is called a multi-set topology on M if τ satisfies the following properties.

1. The m-set M and the empty m-set $\emptyset$ are in τ.
2. the union of elements of any sub collection of τ is in τ.
3. the intersection of elements of any finite sub collection of τ is in τ.

The pair (M, τ) is called multi-topological space (for short M-topological space). The elements of τ are called open msets.

Definition 2.19:[6] Let (M, τ) be an M-topological space and N is a sub mset of M. The collection $\tau_N = \{N \cap U : U \in \tau\}$ is an M-topology on N denoted by $(N, \tau_N)$is called a subspace of (M, τ).

Definition 2.20:[5,6,8,9] A submset N of an M-topological space M in $[X]^w$ is said to be closed m-set if the m-set $M \ominus N$ is open.

Theorem 2.21:[7,6] Let (M, τ) be an M-topological space. Then the following conditions hold.

1. The m-set M and empty m-set $\emptyset$ are closed m-sets.
2. Arbitrary intersection of closed m-sets is a closed m-set.
3. Finite union of closed m-sets is a closed m-set.

Definition 2.22:[5,6]Let $(M, \tau)$and (N, τ^*) be two M-topological spaces .A function $f : (M, \tau) \to (N, \tau^*)$ is said to be m-continuous if the inverse image of τ^* -open mset is τ -open mset .

Definition 2.23:[5,6] Given a submset N of M-topological space M in $[X]^w$.

1. The interior of N is defined as the union of all open msets contained in N and is denoted by Int(N). i.e. $C_{Int(N)}(x) = C_{\cup G}(x) : G$ is an open mset, $G \subseteq N$.
2. The closure of N is defined as the intersection of all closed msets containing N and is denoted by Cl(N). i.e. $C_{CI(N)}(x) = C_{\cap F}(x) : F$ is an closed mset, $F \supseteq N$,

Theorem 2.24:[7,6] Let N be a subspace of an M-topological space M in $[X]^w$ and A be a submset of an mset N and $Cl(A)$ denote the closure of an mset A in M. Then the closure of an mset A in N equals $Cl(A) \cap N$.

Theorem 2.25:[7,6] If M_1, M_2 are submsets of the M-topological space M in $[X]^w$, then the following properties hold:

1. $C_{Int(M_1 \cap M_2)}(x) = Min\{C_{Int(M_1)}(x),\ C_{Int(M_2)}(x)\}$.
2. $C_{Cl(M_1 \cup M_2)}(x) = Max\{C_{Cl(M_1)}(x),\ C_{Cl(M_2)}(x)\}$.

Definition 2.26:[5,13,14,15]Suppose we are given a finite nonempty set U of objects ,called the universe ,,and R is a binary relation defined on U.WE list the properties that are of interest in the theory of rough sets (Pawlak 1982,1991),let $A, B \subset U$

1. $\underline{R}(A) = (\overline{R}(A^c))^c$, $\overline{R}(A) = (\underline{R}(A^c))^c$
2. $\underline{R}(U) = U$, $\overline{R}(\phi) = \phi$
3. $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$, $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$
4. $\underline{R}(A \cup B) \supset \underline{R}(A) \cup \underline{R}(B)$, $\overline{R}(A \cap B) \subset \overline{R}(A) \cap \overline{R}(B)$

5. $A \subset B \Rightarrow \underline{R}(A) \subset \underline{R}(B), \overline{R}(A) \subset \overline{R}(B)$
6. $\underline{R}(\phi) = \phi \quad , \quad \overline{R}(U) = U$
7. $\underline{R}(A) \subset A \quad , \quad \overline{R}(A) \supset A$
8. $A \subset \underline{R}(\overline{R}(A) \quad , \quad A \supset \overline{R}(\underline{R}(A)$
9. $\underline{R}(A) \subset \underline{R}(\underline{R})(A) \quad , \quad \overline{R}(A) \supset \overline{R}(\overline{R})(A)$
10. $\underline{R}(A) \subset \overline{R}(A) \quad , \quad \overline{R}(\underline{R}(A) \subset \underline{R}(A)$

3. Cartesian Product

In this section we study the cartesian product of two multi-sets , the mset relations and some of its properties.

Definition 3.1:[8] Let M_1 and M_2 be two msets drawn from a set X, then the Cartesian product of M_1 and M_2 is defined as $M_1 \times M_2 = \{(m/x, n/y)/mn : x \in^m M_1,\ y \in^n M_2\}$.

Proposition 3.2: Let M_1 and M_2 be two msets drawn from a set X, then

1. $M_1 \times \phi = \phi \times M_1 = \phi$.
2. $M_1 \times (M_2 \cup M_3) = (M_1 \times M_2) \cup (M_1 \times M_3)$.
3. $(M_2 \cup M_3) \times M_1 = (M_2 \times M_1) \cup (M_3 \times M_1)$.
4. $M_1 \times (M_2 \cap M_3) = (M_1 \times M_2) \cap (M_1 \times M_3)$.
5. $(M_2 \cap M_3) \times M_1 = (M_2 \times M_1) \cap (M_3 \times M_1)$.

Proof

1. Trivial.
2. $M_1 \times (M_2 \cup M_3) = \{(m/x, n/y)/mn : x \in^m M_1,\ y \in^n M_2 \cup M_3\}$.
$= \{(m/x, n/y)/mn : x \in^m M_1,\ y \in^n M_2 \cup M_3\}$.
$= \{(m/x, n/y)/mn : m \leq C_{M_1}(x),\ n \leq \max\{C_{M_2}(y), C_{M_3}(y)\}\}$.
$= \{(m/x, n/y)/mn : m \leq C_{M_1}(x),\ (n \leq C_{M_2}(y)\ \ or\ \ n \leq C_{M_3}(y))\}$.
$= \{(m/x, n/y)/mn : x \in^m M_1,\ y \in^n M_2\} \cup \{(m/x, n/y)/mn : x \in^m M_1,\ y \in^n M_3\}$.
$= (M_1 \times M_2) \cup (M_1 \times M_3)$.

3. $(M_2 \cup M_3) \times M_1 = \{(m/x, n/y)/mn : x \in^m M_2 \cup M_3,\ y \in^n M_1\}$.
$= \{(m/x, n/y)/mn : m \leq \max\{C_{M_2}(x), C_{M_3}(x)\},\ n \leq C_{M_1}(y)\}$.
$= \{(m/x, n/y)/mn : (m \leq C_{M_2}(x)\ \ or\ \ m \leq C_{M_3}(x)),\ \ n \leq C_{M_1}(y)\}$.
$= \{(m/x, n/y)/mn : (m \leq C_{M_2}(x), n \leq C_{M_1}(y)) or\ \ (m \leq C_{M_3}(x),\ n \leq C_{M_1}(y))\}$.
$(M_2 \times M_1) \cup (M_3 \times M_1)$.

4. $M_1 \times (M_2 \cap M_3)\ M_1 \times (M_2 \cap M_3) = \{(m/x, n/y)/mn : x \in^m M_1,\ y \in^n M_2 \cap M_3\}$.
$= \{(m/x, n/y)/mn : m \leq C_{M_1}(x),\ n \leq \min\{C_{M_2}(y), C_{M_3}(y)\}\}$.
$= \{(m/x, n/y)/mn : m \leq C_{M_1}(x),\ (n \leq C_{M_2}(y)\ \ and\ \ n \leq C_{M_3}(y))\}$.
$= \{(m/x, n/y)/mn : (m \leq C_{M_1}(x), n \leq C_{M_2}(y)) and\ \ (m \leq C_{M_1}(x),\ n \leq C_{M_3}(y))\}$.
$= (M_1 \times M_2) \cap (M_1 \times M_3)$.

the rest of the proof is similar

Proposition 3.2: Let R ,R_1, R_2 be a mset relations on M ,then

1. R is symmetric $\Leftrightarrow R^{-1}$ is symmetric .
2. R is transitive $\Leftrightarrow R^{-1}$ is transitive .
3. R_1, R_2 are reflexive $\Rightarrow R_1 \cup R_2$ is reflexive.
4. R_1, R_2 are reflexive $\Leftrightarrow R_1 \cap R_2$ is reflexive.
5. R_1, R_2 are symmetric $\Rightarrow R_1 \cup R_2, R_1 \cap R_2$ are symmetric.

Proof

1. R is symmetric $\Leftrightarrow m/xRn/y \Rightarrow n/yRm/x$
   $\Leftrightarrow n/yR^{-1}m/x \Rightarrow m/xR^{-1}n/y$
   $\Leftrightarrow R^{-1}$ is symmetric .
2. Let R be transitive then, if $m/xRn/y, n/yRk/z \Rightarrow m/xRk/z$ then
   $n/yR^{-1}m/x, k/zR^{-1}n/y \Rightarrow k/zR^{-1}m/x$ then R^{-1} is transitive.
3. 1 and 2 are straightforwards
4. $m/xR_1 \cup R_2n/y \Rightarrow m/xR_1n/y$ or $m/xR_2n/y$
   $\Rightarrow n/yR_1m/x$ or $n/yR_2m/x$
   $\Rightarrow n/yR_1 \cup R_2m/x$
   $\Rightarrow R_1 \cup R_2$ is symmetric
5. $m/xR_1 \cap R_2n/y \Rightarrow m/xR_1n/y$ and $m/xR_2n/y \Rightarrow n/yR_1m/x$ and $n/yR_2m/x$
   $\Rightarrow n/yR_1 \cap R_2m/x \Rightarrow R_1 \cap R_2$ is symmetric

But the converse is not true as we show in the following :

Example3.3 Let$f : X \rightarrow Y$,$M = \{3/x, 5/y, 3/z, 7/r\}$, $R_1 = \{(3/x, 3/x)/9, (5/y, 5/y)/25\}$ is not reflexive and ,$R_2 = \{(3/z, 3/z)/9, (7/r, 7/r)/49, (3/x, 5/y)/15\}$ is not reflexive

But the relation $R_1 \cup R_2 = \{(3/x, 3/x)/9, (5/y, 5/y)/25, (3/z, 3/z)/9, (7/r, 7/r)/49, (3/x, 5/y)/15\}$is reflexive .
$R_3 = \{(3/x, 5/y)/15, (3/x, 7/r)/21, (5/y, 3/x)/15, (7/r, 5/y)/35\}$ is not symmetric also ,
the relation $R_4 = \{(5/y, 7/r)/35, (7/r, 3/x)/21\}$ is not symmetric .

But the relation $R_3 \cup R_4 = \{((3/x, 5/y)/15, (3/x, 7/r)/21, (5/y, 3/x)/15, (7/r, 5/y)/35, (5/y, 7/r)/35, (7/r, 3/x)/21\}$is symmetric .
the relation $R_5 = \{(3/x, 3/z)/9, (3/z, 3/x)/9, (7/r, 3/x)/21, (3/x, 5/y)/15\}$ is not symmetric ,also ,$R_6 = \{(3/x, 3/z)/9, (3/z, 3/x)/9, (3/z, 5/y)/15\}$ is not symmetric .
But the relation $R_5 \cap R_6 = \{((3/x, 3/z)/9, (3/z, 3/x)/9\}$is symmetric .

4. Multi-set function

In this section we study the concept of multi-set function and we give a new definitions of the image , the inverse image of a multi-set and study of its properties .
Definition 4.1:[5,6] An mset relation f is called an mset function if for every element m/x in domain f, there is exactly one n/y in Ran f such that $(m/x, n/y)$ is in f with the pair occurring as the product of $C_1(x, y)$ and $C_2(x, y)$.

Definition 4.2:[6,5]A mset function f is called one-one (injective) if no two elements in Dom f have the same image under f with $C_1(x, y) \leq C_2(x, y), \quad forall x, y$

Definition 4.3:[8,5]A mset function f is called onto (surjective) if no two elements in Ran f equal to co-domain f and $C_1(x,y) \geq C_2(x,y), \quad forall x, y$

Definition 4.4:[5,6]A mset function f is called one-to-one and onto (bijective) if it is injective and surjective with $C_1(x,y) = C_2(x,y), \quad forall x, y$

Definition 4.5: Let R be an mset relation defined on M and $m/x \in M$the R- relative mset of m/x is the set of all n/y in M such that there exist some k such that k/x R n/y i.e $R(m/x) = \{n/y : \exists k \in N, k/xRn/y\}$

Example 4.6 Let$M = \{5/a, 4/b, 4/c, 3/d\}$ and N = $\{7/x, 5/y, 6/z, 4/w\}$ be two msets .Let $f : M \rightarrow N$given by $f = \{(5/a, 5/y)/25, (4/b, 6/z)/24, (4/c, 4/w)/16, (3/d, 6/z)/18\}$, $A_1 = \{2/a, 2/b, 3/c\}$, $A_2 = \{5/a, 2/b, 3/c\}$ then, $f(A_1) = \{5/y, 6/z, 4/w\}$,$f(A_2) = \{5/y, 6/z, 4/w\}$.

From the above example we show that the image of $A_1$and A_2 are the same although the two msets are different . then, the definition of the image of mset under a mset function does not depend on the multiplicity of the elements so it is not good extension .So,we can define the image of msetas following :

Definition 4.7: Let M_1 be a mset drawn from a set X ,and M_2be a mset drawn from a set Y . Let $f : M_1 \rightarrow M_2$ be a mset function from M_1 to M_2 . Let A be a submset of M_1, the image of A under the function f denoted by $f(A)$ where $f(A) : Y \rightarrow \mathbf{N}$ defined by

$C_{f(A)}(y)) = min\{M_2(y), sup\{C_A(x) : f(x) = y\}\}$ if $f^{-1}(y) \neq \phi$
$= 0 \quad otherwise$

Example4.8 : Let $f : M \rightarrow N$as defined in the above example we show the images of$A_1$and A_2 are $f(A_1) = \{2/y, 2/z, 3/w\}$,$f(A_2) = \{5/y, 2/z, 3/w\}$ are not the same . In the following we study the properties of the mset function

Proposition 4.9: Let$f : M_1 \rightarrow M_2$be a mset function andA_1, A_2 be submsets ofM_1 .Then ,

1. $A_1 \subseteq A_2 \Rightarrow f(A_1) \leq f(A_2)$
2. $f(A_1) \cup f(A_2) = f(A_1 \cup A_2)$
3. $f(A_1 \cap A_2) \leq f(A_1) \cap f(A_2)$
4. $f(A_1) \oplus f(A_2) = f(A_1 \oplus A_2)$
5. $f(A_1) \ominus f(A_2) \leq f(A_1 \ominus A_2)$

Proof

1. $C_{f(A_1)}(y)) = min\{M_2(y), sup\{C_{A_1}(x) : f(x) = y\}\}$ .Since , $C_{A_1}(x) \leq C_{A_2}(x) \forall x \in X$,then we have$C_{f(A_1)}(y)) = min\{M_2(y), sup\{C_{A_1}(x) : f(x) = y\}\} \leq min\{M_2(y), sup\{C_{A_2}(x) : f(x) = y\}\} = C_{f(A_2)}(y))$.Thus , we have $f(A_1) \subseteq f(A_2)$
2. Since $A_1, A_2 \subseteq A_1 \cup A_2$then, $f(A_1), f(A_2) \subseteq f(A_1 \cup A_2)$

   $C_{f(A_1 \cup A_2)}(y) = min\{C_{M_2}(y), sup\{C_{(A_1 \cup A_2)}(x) : f(x) = y\}\} = min\{C_{M_2}(y), sup\{C_{(A_1 \cup A_2)}(x) : f(x) = y\}\} = min\{C_{M_2}(y), sup\{\max\{C_((A_1)(x), C_((A_2)(x)\}\} : f(x) = y\}\} = min\{C_{M_2}(y), \max(sup\ \{C_((A_1)(x) : f(x) = y\}, sup\ \{C_((A_2)(x) : f(x) = y\})\} = \max(min\{C_{M_2}(y), sup\ \{C_((A_1)(x) : f(x) = y\}, min\{C_{M_2}(y), sup\ \{C_((A_2$ $f(x) = y\}) = \max\{C_{f(A_1)}(y), C_{f(A_2)}(y)\}$.Thus ,$C_{f(A_1 \cup A_2)}(y) = \max\{C_{f(A_1)}(y), C_{f(A_2)}(y)\}$ .Consequently , we have $f(A_1) \cup f(A_2) = f(A_1 \cup A_2)$

3. $C_{f(A_1\cap A_2)}(y) = min\{C_{M_2}(y), sup\{C_{(A_1\cap A_2)}(x) : f(x) = y\}\} = min\{C_{M_2}(y), sup\{\min\{C_(A_1)(x), C_(A_2)(x)\} : f(x) = y\}\} \leq min\{C_{M_2}(y), \min\{\sup\{C_(A_1)(x) : f(x) = y\}, \sup\{C_(A_2)(x)\} : f(x) = y\}\} \leq min\{\min\{C_{M_2}(y), \sup\{C_(A_1)(x) : f(x) = y\}, \min\{\{C_{M_2}(y), , \sup\{C_(A_2)(x)\} : f(x) = y\}\} \leq \min\{C_{f(A_1)}(y), C_{f(A_2)}($

   $C_{f(A_1)\cap f(A_2)}(x)$.Thus $f(A_1 \cap A_2) \leq f(A_1) \cap f(A_2)$
4. $C_{f(A_1\oplus A_2)}(y) = min\{C_{M_2}(y), sup\{C_{(A_1\oplus A_2)}(x) : f(x) = y\}\} = min\{C_{M_2}(y), sup\{C_{A_1}(x)+C_{A_2}(x) : f(x) = y\}\} = min\{C_{M_2}(y), sup\{C_{A_1}(x) : f(x) = y\}\}+\sup\{C_{A_2}(x) : f(x) = y\}\} = min\{C_{M_2}(y), sup\{C_{A_1}(x) : f(x) = y\}\}+\min\{C_{M_2}(y), \sup\{C_{A_2}(x) : f(x) = y\}\} = C_{f(A_1)}(y)+C_{f(A_2)}(y)$.Consequently ,$f(A_1)\oplus f(A_2) = f(A_1 \oplus A_2)$.
5. $C_{f(A_1\ominus A_2)}(y) = min\{C_{M_2}(y), sup\{C_{(A_1\ominus A_2)}(x) : f(x) = y\}\} = min\{C_{M_2}(y), sup\{\max\{C_{A_1}(x) - C_{A_2}(x), 0\} : f(x) = y\}\} \geq \max\{C_{f(A_1)}(y)-C_{f(A_2)}(y), 0\}$.Consequently ,$f(A_1)\ominus f(A_2) \leq f(A_1\ominus A_2)$

But the converse is not true.

Example 4.10 Let$M = \{8/x, 7/y, 1/z\}$, $N = \{6/a, 9/b\}$,and $f = \{(8/x, 6/a)/8, (7/y, 9/b))/7, (11/z, 6/a)/11\}$ be a mset function ,$M_1 = \{5/x, 3/y\}$, $M_2 = \{8/Z\}$,$M_1 \cap M_2 = \phi$,$f(M_1) = \{5/a, 3/b\}$,$f(M_2) = \{4/a\}$,$f(M_1)\cap f(M_2) = \{5/a\}$, $f(M_1)\ominus f(M_2) = \{3/b\}$,$M_1\ominus M_2 = \{5/x, 3/y\}$, $f(M_1\ominus M_2) = \{5/a, 3/b\}$, $f(M_1 \ominus M_2) \supseteq f(M_1) \ominus f(M_2)$, $f(M_1) \cap f(M_2) \subseteq f(M_1 \cap M_2)$

Example 4.11 Let$M = \{5/a, 4/b, 4/c, 3/d\}$ and N $= \{7/x, 5/y, 6/z, 4/w\}$ be two msets .Let $g : M \to N$given by $g = \{(5/a, 7/y)/35, (4/b, 7/x)/28, (4/c, 6/z)/24, (3/d, 4/w)/12\}$, $B_1 = \{5/y, 6/z, 4/w\}$, $B_2 = \{2/y, 3/z, 2/w\}$ then,$f^{-1}B_1 = \{5/a, 4/c, 3/d\}$and $f^{-1}B_2 = \{5/a, 4/c, 3/d\}$ are the same although the two msets are different . then, the definition of the inverse image of mset under a mset function does not depend on the multiplicity of the elements so it is not good extension .So,we can define the inverse image of mset as following :

Definition 4.12: Let$f : X \to Y$ be a function from X into Y and let B be submset ofM_2,then ,the map $f^{-1} : M_1 \to M_2$ defined by $C_{f^{-1}B}(x) = min\{C_{M_1}(x), C_B(f(x))\}$is called the inverse image of a mset B of M_2 ,whereM_1 is a mset on X andM_2 be a m-subset on Y

Example4.13 Let$M = \{5/a, 4/b, 4/c, 3/d\}$ and N $= \{7/x, 5/y, 6/z, 4/w\}$ be two m-sets .Let $g : M \to N$given by $g = \{(5/a, 7/y)/35, (4/b, 7/x)/28, (4/c, 6/z)/24, (3/d, 4/w)/12\}$, $B_1 = \{5/y, 6/z, 4/w\}$, $B_2 = \{2/y, 3/z, 2/w\}$ then,$f^{-1}B_1 = \{5/a, 4/c, 3/d\}$and $f^{-1}B_2 = \{2/a, 3/c, 2/d\}$are not the same . In the following we study the properties of the inverse of mset function

Proposition 4.14: Let$f : M_1 \to M_2$be a mset function andB_1, B_2 be submsets ofM_2 .Then ,

1. $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$
2. $f^{-1}(B_1) \cup f^{-1}(B_2) = f^{-1}(B_1 \cup B_2)$
3. $f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2)$
4. $f^{-1}(B_1) \oplus f^{-1}(B_2) = f^{-1}(B_1 \oplus B_2)$
5. $f^{-1}(\phi) = \phi$,$f^{-1}((M_2) = M_1$

Proof

1. $C_{f^{-1}(B_1)}(x) = \min\{C_{M_1}(x), C_{B_1}(f(x))\} \leq \min\{C_{M_1}(x), C_{B_2}(f(x))\} = C_{f^{-1}(B_1)}(x)$
2. $C_{f^{-1}(B_1\cup B_2)}(x) = \min\{C_{M_1}(x), C_{B_1\cup B_2}(f(x))\} = \min\{C_{M_1}(x), \max\{C_{B_1}(f(x)), C_{B_2}(f(x))\}\} = \max\{C_{f^{-1}(B_1)}(x), C_{f^{-1}(B_2)}(x)\} = C_{f^{-1}(B_1)\cup f^{-1}(B_2)}(x)$

3. $C_{f^{-1}(B_1 \cap B_2)}(x) = \min\{C_{M_1}(x), C_{B_1 \cap B_2}(f(x))\} = \min\{C_{M_1}(x), \min\{C_{B_1}(f(x)), C_{B_2}(f(x))\}\}$

   $= \min\{C_{f^{-1}(B_1)}(x), C_{f^{-1}(B_2)}(x)\} = C_{f^{-1}(B_1) \cap f^{-1}(B_2)}(x)$

.The others are similar

Proposition 4.15: Let$f : M \to N$be a mset function and let$A$ be submset of M , B be submset of N ,then

1. $f(f^{-1}(B) \subseteq B$,and equality holds if $f$is onto.

2. $f^{-1}(f(A)) \supseteq A$,and equality holds if $f$is one-to-one.

Proof

1. $C_{f(f^{-1}(B)}(y) = \min\{C_N(y), \sup\{C_{f^{-1}(B}(x) : f(x) = y\}\} \leq \min\{C_N(y), C_B(y)\} = C_B(y)$

.The other are similar.

5. Multi-order relation

In this section we study multi-order relation and some of its properties

Definition 5.1:A mset relation R on M is said to be multi-pre-order relation if it is reflexive and transitive .

Definition 5.2: A mset relation R on M is said to be multi-antisymmetric relation if it is $(m/x, n/y) \in R, (n/y, m/x) \in R \Rightarrow m/x = n/y$

Definition 5.3:A mset relation R on M is said to be multi-partial order relation if it is reflexive , anti-symmetric and transitive . The pair (M, R) is said to be multi-partially ordered set denoted for short (m - poset).

Definition 5.4:A multi-totally ordered set is a m - poset relation (M, R) in which for any two multi-points m/a , n/b in M are comparable , that is $\forall m/x, n/y \in M \Rightarrow (m/x, n/y) \in R$ or $(n/y, m/x) \in R$

Definition 5.5:A mset relation D on M is said to be multi-directed relation if it is reflexive , transitive and satisfies the following condition $\forall m/x, n/y \in M \exists r/z \in M$ s . t. $(r/z, m/x) \in D, (r/z, n/y) \in D$

The pair (M, D) is said to be multi-directed set or m - directed system .

Definition 5.5: Let M be a mset drawn from a set X ,A be a submset of M .A multi-point n/x is said to be m- quasi-coincident of A Iff $n + C_A(x) > C_M(x)$ and denoted by n/xqA

Definition 5.6: Let M be a mset drawn from a set X ,A and B be two msets of M .A multiset A is said to be m- quasi-coincident of B Iff $\exists x \in X \quad s.t C_A(x) + C_B(x) > C_M(x)$ and denoted by $A \quad qB$

Proposition 5.7: Let$f : M_1 \to M_2$be a mset function and B be a mset on M_2, n/b a multi-point on M_1 then we have $f(n/b)qB \Rightarrow n/bqf^{-1}(B)$

Proof Since $f(n/b)qB \Rightarrow n + C_B(f(b)) > C_{M_2}(f(b))$ then , $n + C_{f^{-1}(B)}(b) > C_{f^{-1}(M_2)}(b)$ that is $n + C_{f^{-1}(B)}(b) > C_{(M_1)}(b)$.Hence we have $n/bqf^{-1}(B)$

Definition 5.8: Let(M, R) be multi-partially ordered set . A mset A of M is said to be :

1. q- multi-increasing if $\forall m/a \in A, n/b \in M, (m/a, n/b) \in R$,then n/bqA
2. q-multi-decreasing if $\forall m/a \in A, n/b \in M, (n/b, m/a) \in R$,then n/bqA

.

Example 5.9: Let X $= \{a, b, c\}$,M $= \{4/a, 5/b, 4/c, 6/d\}$, and
let R be a mset relation on M defined by $m/aRn/b \Leftrightarrow a \neq b$,then
$R = \{(4/a, 5/b), (4/a, 4/c), (4/a, 6/d), (5/b, 4/a), (5/b, 4/c), (5/b, 6/d), (4/c, 4/a)(4/c, 5/b), (4/c, 6/d), (6/d, 4/a), (6/d, 5/b)$
then A $= \{3/a, 4/b, 2/d, 3/c\}$is a q- m-increasing mset and B $= \{3/a, 4/b, 1/d, 3/c\}$is a m-decreasing mset

Example 5.10: Let X $= \{a, b, c\}$,M $= \{3/a, 3/b, 4/c\}$,and let R be a m-set relation on M defined by $m/aRn/b \Leftrightarrow a \neq b$ and $m < n$, then $R = \{(3/a, 4/c), (3/b, 4/c)\}$, then A $= \{3/a, 2/b, 1/c\}$is a q- m-increasing m-set .

Proposition 5.11: Let(M, R) be multi-partially ordered set and A ,B be m-subsets of M such that A is q- m-increasing mset and B is q- m-decreasing of M .Then A^c is q- m-decreasing mset and B^c is q-m- increasing mset , where A^c ,B^c are complements of A and B respectively .

Proof Let A be q-m-increasing mset and let $m/a \in A^c, n/b \in M, (n/b, m/a) \in R$.If $n/b\overline{q}A^c$,then $n/b \in A$,A is q-m-increasing which implies m/aqA.Then , $m/a \notin A^c$ which is contridiction. Hence A^c is q-m-decreasing mset of M .

Let B be q-m- decreasing mset and let $m/a \in B^c, n/b \in M, (m/a, n/b) \in R$. If $n/b\overline{q}B^c$,then $n/b \in B$,B is q-m- decreasing which implies m/aqB.Then , $m/a \notin B^c$ which is contridiction . Hence B^c is increasing mset of M .

Proposition 5.12: Let(M, R) be multi-partially ordered set .The intersection of two q-m-increasing (resp. q-m-decreasing) msets is q-m-increasing (resp. q-m-decreasing) msets ,also The union of two q-m-increasing (resp. q-m-decreasing) msets is q-m-increasing (resp. q-m-decreasing) msets

Proof Let A and B be q-m-increasing msets and $m/a \in A \cap B, (m/a, n/b) \in R \Rightarrow m/a \in A$ and $m/a \in B, (m/a, n/b) \in R$ then ,we have $n/bqA, n/bqB$,implies $n + min\{C_A(b), C_B(b)\} > C_M(b) \Rightarrow n + C_{A\cap B}(b) > C_M(b)$ .Hence ,we have $n/b \ q \ A \cap B$,consequently we have $A \cap B$ is q-m- increasing .Similarly we can show that the union of two q-m-increasing msets is also q-m-increasing mset

Definition 5.13: Let (M, R), $(N, R^\star)$ be two m-posets. A mapping $f : (M, R) \to (N, R^\star)$ is said

1. q-m-increasing (q- m-decreasing) if $\forall m/a, n/b \in M, (m/a, n/b) \in R \Rightarrow (f(m/a), f(n/b)) \in R^\star$ ($(f(n/b), f(m/a)) \in R^\star$)
2. m-order embedding if $\forall (m/a, n/b) \in R \Leftrightarrow (f(m/a), f(n/b)) \in R^\star$
3. m-order reverse embedding if $\forall (m/a, n/b) \in R \Leftrightarrow (f(n/b), , f(m/a)) \in R^\star$

Example5.14 : Let$M = \{5/a, 4/b, 4/c, 3/d\}, N = \{7/x, 5/y, 6/z, 4/w\}$ and $f : (M, R) \to (N, R)$be a m-set function where $f = \{(5/a, 5/y)/25, (4/b, 4/z)/16, (4/c, 4/w)/16, (3/d, 3/z)/9\}$and R be a m-set relation on M defined by $m/aRn/b$ iff $m < n$ then , f is q- m-increasing function from M to N .

.**Proposition 5.15:**Let $f : (M, R) \to (N, R^\star)$ be a mapping and f is q- m-increasing map then the inverse image of q-m-increasing (q- m-decreasing) mset of M is q- m- increasing (q-m-decreasing) of M

Proof Let B be q-m increasing mset of N and let m/a $\in f^{-1}(B)$,n/b $\in M$such that $(m/a, n/b) \in R$.Then , $f(m/a) \in B$,$(f(m/a), f(n/b)) \in R^\star$,since B is q- m- increasing then, $f(n/b)qB \Rightarrow n/bqf^{-1}(B)$ consequently $f^{-1}(B)$ is q-m- increasing m-subset of M.Similarly , we can prove that the inverse image of q- m-decreasing mset of M is q-m-decreasing of M .

6. m-topological ordered spaces

In this section we study multi-topological order space ,some of separation axioms of its and study some of its relation between them

Definition 6.1: A m-topological ordered space on M is the triple $(M, \tau, R)$where (M, R) is a m-poset and $(M, \tau)$is m-topological space

Definition 6.2: A m-topological ordered space$(M, \tau, R)$is said to be :

1. m- LowerT_1 ordered space iff $\forall m/a, n/b \quad s.t. m/a\overline{R}n/b$,$\exists$ q-m-increasing open mset u such that $m/a \in u, n/b \notin u$
2. m- upper T_1 ordered space iff $\forall m/a, n/b \quad s.t. m/a\overline{R}n/b$,$\exists$ q-m-decreasing open mset u such that $n/b \in u, m/a \notin u$
3. m- T_0 ordered space if it is m-lower T_1 or m-upperT_1ordered space .
4. m- T_1 ordered space if it is m-lower T_1 and m-upperT_1ordered space .
5. m- T_2 ordered space if $\forall m/a, n/b \quad s.t. m/a\overline{R}n/b$,$\exists$ q-m-increasing open mset u and q- m- decreasing open mset v , such that $m/a \in u, n/b \in v$ and $u \cap v = \phi$

.

Proposition 6.3: Let(M, τ_1, R),(N, τ_2, R^{*}) be two m-topological ordered spaces and $f : (M, \tau_1, R) \rightarrow (N, \tau_2, R^{\star})$be a m-continuous and m-increasing map from M onto N .Then,if $(N, \tau_2, R^{\star})$ is m- LowerT_1 then , (M, τ_1, R) m- LowerT_1.Also.if $(N, \tau_2, R^{\star})$ is m- upperT_1 then , (M, τ_1, R) m- upperT_1

Proof Let $m/a\overline{R} \quad n/b$,since f is m-increasing then $f(m/a)\overline{R^{\star}}f(n/b)$.Since$(N, \tau_2, R^{\star})$ m- LowerT_1, then $\exists$q- m-increasing open m-subset u of N such that $f(m/a) \in u, f(n/b) \notin u \Rightarrow \exists f^{-1}(u)$ is q-m-increasing open m -subset of M and $m/a \in f^{-1}(u), n/b \notin f^{-1}(u)$ .Consequantly , the space(M, τ_1, R) is m-lower T_1 .Similarly we can prove upper case

Proposition 6.4: Let (M , R)be m-poset .The class of all q-m-increasing (q-m-decreasing) msets forms a m-topology on M ,it is denoted by τ_{inc} (τ_{dec})

Proof

1. ϕ,M are q- m-increasing msets
2. Let A , B be q- m-increasing msets ,and let $m/a \in A \cap B$, $n/b \in M$, $(m/a, n/b) \in R \Rightarrow m/a \in A$ and $m/a \in B$. since , A , B are q-m-increasing then,we have $n/b \in A \cap B$.Hence , $A \cap B$ is q-m-increasing .
3. $m/a \in \cup_{\alpha} A_{\alpha}$, $n/b \in M$, $(m/a, n/b) \in R$.
   If n/b $\overline{q} \cup_{\alpha} A_{\alpha} \Rightarrow n/b\overline{q}A_{\alpha}\forall\alpha \Rightarrow n/b \notin A_{\alpha}\forall\alpha$ which is contradiction for A_{α} is increasing $\forall\alpha$

Similarly we can show that the class τ_{dec} is m- topology .

Definition 6.5: Let R be an m-binary relation on M . The after m- set of m/a $\in$ M is denoted by m/a R and it defined by $m/aR = \{n/b : (m/a, n/b) \in R\}$.Also , the fore m-set of m/a $\in$ M is denoted by R m/a and it defined by $Rm/a = \{n/b : (n/b, m/a) \in R\}$

Proposition 6.6: Let R be an m-binary relation on M .Then , the class $\tau_R^{\star} = \{A \subseteq M : m/aR \subseteq$

$A \ \forall \ m/aqA\}$ is an m-topology on M

Proof

1. $\phi, M \in \tau_R^\star$
2. Let $A_1, A_2 \in \tau_R^\star$. Let $m/aqA_1 \cap A_2 \Rightarrow m/aqA_1, m/aqA_2$.Then , $m/aR \subseteq A_1, m/aR \subseteq A_2 \Rightarrow m/aR \subseteq A_1 \cap A_2$.Hence , $A_1 \cap A_2 \in \tau_R^\star$
3. Let $A_\alpha \in \tau_R^\star \ \forall\alpha$ and let $m/a \ \ q \cup_\alpha A_\alpha$then , $m/a \ \ qA_\alpha$for some α .then,$m/aR \subseteq A_\alpha$ for some α.Hence ,$m/aR \subseteq \cup_\alpha A_\alpha$,consequently ,$\cup_\alpha A_\alpha \in \tau_R^\star$.Thus ,$\tau_R^\star$is an m- topology

Definition 6.7:A triple $(M, \tau_R^\star, \rho)$ is called an m- order topological approximation space (denoted by m-OTAS) , where $\tau_R^\star = \{A \subseteq M : m/aR \subseteq A \ \ \forall \ m/aqA\}$,and ρ is m- partially order relation .

Definition 6.8:A triple $(M, \tau_R^\star, \rho)$ be an m-OTAS and A be submset of M .The m-lower ,m-upper approximations ,m-boundary region and m-accuracy respectively are given by :
$\underline{R}_{\tau_R^\star - inc}(A) = \cup\{G \in \tau_R^\star : Gis \ \ q - m - increasing, G \subset A\}$

$\underline{R}_{\tau_R^\star - dec}(A) = \cup\{G \in \tau_R^\star : Gis \ \ q - m - decreasing, G \subset A\}$

$\overline{R}_{\tau_R^{\star c} - inc}(A) = \cap\{F \in \tau_R^{\star c} : Fis \ \ q - m - increasing, A \subset F\}$

$\overline{R}_{\tau_R^{\star c} - dec}(A) = \cap\{F \in \tau_R^{\star c} : Fis \ \ q - m - decreasing, A \subset F\}$

$BN_{\tau_R^\star - inc}(A) = \overline{R}_{\tau_R^\star - inc}(A) \setminus \underline{R}_{\tau_R^\star - inc}(A)$
$BN_{\tau_R^\star - dec}(A) = \overline{R}_{\tau_R^\star - dec}(A) \setminus \underline{R}_{\tau_R^\star - dec}(A)$
$\alpha^{\tau_R^\star - inc}(A) = \frac{|\underline{R}_{\tau_R^\star - inc}(A)|}{|\overline{R}_{\tau_R^\star - inc}(A)|}$
$\alpha^{\tau_R^\star - dec}(A) = \frac{|\underline{R}_{\tau_R^\star - dec}(A)|}{|\overline{R}_{\tau_R^\star - dec}(A)|}$

Proposition 6.9: Let$(M, \tau_R^\star, \rho)$ be an m-OTAS and A ,B be submsets of M .Then ,

1. $\underline{R}_{\tau_R^\star - inc}(A) \subset A \subset \overline{R}_{\tau_R^\star - inc}(A)(\underline{R}_{\tau_R^\star - dec}(A) \subset A \subset \overline{R}_{\tau_R^\star - dec}(A))$
2. $A \subset B \Rightarrow \underline{R}_{\tau_R^\star - inc}(A) \subset \underline{R}_{\tau_R^\star - inc}(B)(\underline{R}_{\tau_R^\star - dec}(A) \subset \underline{R}_{\tau_R^\star - dec}(B)$
3. $A \subset B \Rightarrow \overline{R}_{\tau_R^{\star c} - inc}(A) \subset \overline{R}^{\tau_R^{\star c} - inc}(B)(\overline{R}_{\tau_R^{\star c} - dec}(A) \subset \overline{R}_{\tau_R^{\star c} - dec}(B)$
4. $\overline{R}_{\tau_R^{\star c} - inc}(A \cap B) \subset \overline{R}_{\tau_R^{\star c} - inc}(A) \cap \overline{R}_{\tau_R^{\star c} - inc}(B)$
5. $\overline{R}_{\tau_R^{\star c} - inc}(A \cup B) = \overline{R}_{\tau_R^{\star c} - inc}(A) \cup \overline{R}_{\tau_R^{\star c} - inc}(B)$
6. $\underline{R}_{\tau_R^\star - inc}(A \cup B) \supset \underline{R}_{\tau_R^\star - inc}(A) \cup \underline{R}_{\tau_R^\star - inc}(B)$
7. $\underline{R}_{\tau_R^\star - inc}(A \cap B) = \underline{R}_{\tau_R^\star - inc}(A) \cap \underline{R}_{\tau_R^\star - inc}(B)$

Proof

1. Straightforward
2. Let$A \subseteq B$ then the family $\{G \in \tau_R^\star : Gis \ \ q - m - increasing, G \subset A\} \subset \{G \in \tau_R^\star : Gis \ \ q - m - increasing, G \subset B\}$ which implies that $\cup\{G \in \tau_R^\star : Gis \ \ q - m - increasing, G \subset A\} \subset \cup\{G \in \tau_R^\star : Gis \ \ q - m - increasing, G \subset B\}$,that is$\underline{R}_{\tau_R^\star - inc}(A) \subset A \subset \underline{R}_{\tau_R^\star - inc}(A)$.Similarly ,we can show that $\underline{R}_{\tau_R^\star - dec}(A) \subset A \subset \underline{R}_{\tau_R^\star - dec}(A)$

3. Let$A \subseteq B$ then the family $\{F \in \tau_R^{\star c} : Gis \;\; q-m-increasing, B \subset F\} \subset \{F \in \tau_R^{\star c} : Fis \;\; q-m-increasing, A \subset F\}$ which implies that $\cap\{F \in \tau_R^{\star c} : Gis \;\; q-m-increasing, A \subset F\} \subset \cap\{F \in \tau_R^{\star c} : Fis \;\; q-m-increasing, B \subset F\}$,that is $\overline{R}^{\tau_R^{\star}-inc}(A) \subset A \subset \overline{R}^{\tau_R^{\star}-inc}(B)$.Similarlywe can show that $\overline{R}^{\tau_R^{\star}-dec}(A) \subset A \subset \overline{R}^{\tau_R^{\star}-dec}(B)$
4. Since $A \cap B \subset A$ and $A \cap B \subset B$ We have $\overline{R}^{\tau_R^{c}-dec}(A \cap B) \subset \overline{R}^{\tau_R^{\star}-dec}(A) \cap \overline{R}^{\tau_R^{c}-dec}(B)$.Since $\{F \in \tau_R^{\star c} : Fis \;\; q-m-increasing, A \subset F\} \subset \{F \in \tau_R^{\star c} : Fis \;\; q-m-increasing, A \cap B \subset F\}$ then,we have $\overline{R}^{\tau_R^{\star}-dec}(A) \subset \overline{R}^{\tau_R^{\star}-dec}(A \cap B)$ and also $\overline{R}^{\tau_R^{\star}-dec}(B) \subset \overline{R}^{\tau_R^{\star}-dec}(A \cap B)$.Hence we have $\overline{R}^{\tau_R^{\star}-dec}(A) \cap \overline{R}^{\tau_R^{\star}-dec}(B) = \overline{R}^{\tau_R^{\star}-dec}(A \cap B)$ .Similarly for decreasing case
5. Since $\overline{R}^{\tau_R^{\star}-inc}(A) \cup \overline{R}^{\tau_R^{\star}-inc}(B) = (\cap\{F \in \tau_R^{\star c} : Fis \;\; q-m-increasing, A \subset F\}) \cup (\cap\{F' \in \tau_R^{\star c} : F' is \;\; q-m-increasing, B \subset F'\}) = \cap\{F \cup F' \in \tau_R^{\star c} : F, F' are \;\; q-m-increasing, A \subset FandB \subset F'\} = \cap\{F^\star \in \tau_R^{\star c} : F^\star is \;\; q-m-increasing, A \cup B \subset F^\star\} = \overline{R}^{\tau_R^{\star}-inc}(A \cup B)$
6. Since $A \subset A \cup B$ and $B \subset A \cup B$ then we have $\underline{R}_{\tau_R^{\star}-inc}(A) \cup \underline{R}_{\tau_R^{\star}-inc}(B) \subset \underline{R}_{\tau_R^{\star}-inc}(A \cup B)$
7. Since $\underline{R}_{\tau_R^{\star}-inc}(A) \cap \underline{R}_{\tau_R^{\star}-inc}(B) = (\cup\{G \in \tau_R^{\star} : Gis \;\; q-m-increasing, G \subset A\}) \cap (\cup\{H \in \tau_R^{\star} : Gis \;\; q-m-increasing, H \subset B\}) = \cup\{G \cap H \in \tau_R^{\star} : Gis \;\; q-m-increasing, G \subset AandHis \;\; q-m-increasing, H \subset B\} = \cup\{H' \in \tau_R^{\star} : H' is \;\; q-m-increasing, H' \subset A \cap B\} = \underline{R}_{\tau_R^{\star}-inc}(A \cap B)$

in the following example we find The m-lower ,m-upper approximations and m-accuracy for some msets

Example6.11 : Let$M = \{3/a, 2/b, 4/c, 5/d\}$,R $=\{(3/a, 3/a)/9, (2/b, 2/b)/4, (4/c, 4/c)/16, (5/d, 5/d)/25$

$, (3/a/2/b)/6, (2/b, 1/c), /2, (5/d, 2/c)/10\}$

then we have $\tau_R^{\star} = \{\phi, M, \{3/a, 2/b, 4/c\}, \{2/b, 4/c\}, \{4/c\}, \{5/d, 4/c\}\}$

A	$\underline{R}_{inc}(A)$	$\overline{R}^{inc}(A)$	$\overline{\alpha}^{inc}(A)$	BND_{inc}
$\{3/a, 4/c\}$	$\{4/c\}$	M	4/5	$\{3/a, 2/b, 5/d\}$
$\{3/a, 2/b\}$	ϕ	M	0	$\{3/a, 2/b, 4/c, 5/d\}$
$\{2/d, 4/c\}$	$\{4/c\}$	M	4/5	$\{3/a, 2/b, 5/d\}$
$\{5/d, 4/c\}$	$\{5/d, 4/c\}$	M	1	$\{3/a, 2/b\}$
$\{2/b, 4/c\}$	$\{2/b, 4/c\}$	M	4/5	$\{3/a, 5/d\}$
$\{1/b, 4/c\}$	$\{4/c\}$	M	4/5	$\{3/a, 2/b, 5/d\}$
$\{2/a, 2/b, 4/c\}$	$\{2/b, 4/c\}$	M	4/5	$\{3/a, 5/d\}$
$\{3/a, 2/d\}$	ϕ	M	0	$\{3/a, 2/b, 4/c, 5/d\}$
$\{4/d\}$	ϕ	M	0	$\{3/a, 2/b, 4/c, 5/d\}$
$\{3/a\}$	ϕ	M	0	$\{3/a, 2/b, 4/c, 5/d\}$,
$\{3/c\}$	ϕ	M	0	$\{3/a, 2/b, 4/c, 5/d\}$,
$\{2/b\}$	ϕ	M	0	$\{3/a, 2/b, 4/c, 5/d\}$,

.

Table(1)

From the table we have $\underline{R}_{\tau_R^{\star}-inc}(\{3/a, 4/c\}) = \{4/c\}$,$\underline{R}_{\tau_R^{\star}-inc}(\{3/a, 2/b\}) = \phi$ then
$\underline{R}_{\tau_R^{\star}-inc}(\{3/a, 4/c\}) \cup \underline{R}_{\tau_R^{\star}-inc}(\{3/a, 2/b\}) = \{4/c\}$
but $\underline{R}_{\tau_R^{\star}-inc}(\{3/a, 2/b\} \cup \{3/a, 4/c\}) = \underline{R}_{\tau_R^{\star}-inc}\{3/a, 2/b, 4/c\} = \{3/a, 2/b, 4/c\} \supset \underline{R}_{\tau_R^{\star}-inc}(\{3/a, 2/b\}) \cup \underline{R}_{\tau_R^{\star}-inc}(\{3/a, 4/c\})$

in the following we define a different types of generalized rough msets and illustrate the relationships between different types of generalized definitions of rough multi-set approximations

Definition 6.12:: Let R be any mset relation on a non-empty met M .For any submet $A \subset M$ the lower and the upper mset approximations of A are defined by $\overline{R^*}(A) = \cup\{m/xR : m/xRqA\}$ and $\underline{R^*}(A) = (\overline{R^*}(A^c))^c$ and the m - accuracy is defined by $\alpha^*(A) = \mid \frac{\underline{R^*}(A)}{\overline{R^*}(A)} \mid$

Definition 6.13: Let R be any met relation on a non-empty met M .For any submet $A \subset M$ the lower and the upper mset approximations of A are define as $\underline{R^{**}}(A) = \cup\{m/xR : m/xR \subset A\}$ and $\overline{R^{**}}(A) = (\underline{R^{**}}(A^c))^c$ and the m - accuracy is defined by $\alpha^{**}(A) = \mid \frac{\underline{R^{**}}(A)}{\overline{R^{**}}(A)} \mid$
in the following example we find The m-lower ,m-upper approximations and m-accuracy for some msets

Example 6.14 : Let$M = \{3/a, 2/b, 4/c, 5/d\}$,R =$\{(3/a, 3/a)/9, (2/b, 2/b)/4, (4/c, 4/c)/16, (5/d, 5/d)/25$

$, (3/a/2/b)/6, (2/b, 1/c)/2, (5/d, 2/c)/10\}$

A	$\underline{R^*}(A)$	$\overline{R}^*(A)$	$\alpha^*(A)$	BND^*
$\{3/a, 4/c\}$	$\{2/c\}$	$\{3/a, 2/b, 4/c, 5/d\}$	2/5	$\{3/a, 2/b, 2/c, 5/d\}$
$\{3/a, 2/b\}$	$\{3/a\}$	$\{3/a, 2/b, 1/c\}$	1	$\{2/b, 1/c\}$
$\{2/d, 4/c\}$	ϕ	$\{2/b, 4/c, 5/d\}$	0	$\{2/b, 4/c, 5/d\}$
$\{2/b, 4/c\}$	$\{2/c\}$	M	2/5	$\{3/a, 2/b, 2/c, 5/d\}$
$\{2/d, 3/a\}$	ϕ	$\{3/a, 2/b, , 2/c, 5/d\}$	0	$\{3/a, 2/b, 2/c, 5/d\}$
$\{2/a, 2/b, 4/c\}$	$\{2/c\}$	M	2/5	$\{3/a, 2/b, 2/c, 5/d\}$
$\{4/d\}$	ϕ	$\{5/d, 2/c\}$	0	$\{2/c, 5/d\}$
$\{3/a\}$	$\{3/a\}$	$\{3/a, 2/b\}$	1	$\{2/b\}$
$\{3/c\}$	ϕ	$\{5/d, 4/c\}$	0	$M = \{4/c, 5/d\}$
$\{2/b\}$	ϕ	$\{3/a, 2/b, 1/c\}$	0	$M = \{3/a, 2/b, 1/c\}$

Table(2)

A	$\underline{R^{**}}(A)$	$\overline{R}^{**}(A)$	$\alpha^{**}(A)$	BND^{**}
$\{3/a, 4/c\}$	$\{4/c\}$	M	4/5	$\{3/a, 2/b, 5/d\}$
$\{3/a, 2/b\}$	$\{3/a, 2/b\}$	$\{3/a, 2/b\}$	1	ϕ
$\{2/d, 4/c\}$	$\{4/c\}$	$\{4/c, 5/d\}$	4/5	$\{5/d\}$
$\{2/b, 4/c\}$	$\{2/b, , 4/c\}$	M	4/5	$\{3/a, 5/d\}$
$\{2/d, 3/a\}$	ϕ	$\{3/a, 5/d\}$	0	$\{3/a5/d\}$
$\{2/a, 2/b, 4/c\}$	$\{2/b, 4/c\}$	M	4/5	$\{3/a, 5/d\}$
$\{4/d\}$	ϕ	$\{5/d\}$	0	$\{5/d\}$
$\{3/a\}$	ϕ	$\{3/a\}$	0	$\{3/a\}$
$\{3/c\}$	ϕ	$\{5/d, 4/c\}$	0	$\{4/c, 5/d\}$
$\{2/b\}$	ϕ	$\{3/a, 2/b\}$	0	$\{3/a, 2/b\}$
$\{1/b, 4/c\}$	$\{4/c\}$	M	4/5	$\{3/a, 2/b, 5/d\},$

Table(3)

References

[1] W.D.Blizard ,Multiset theory ,Notre Dame Journal of Logic 30 (1989) , 346 -368 .

[2] T.H.Choe ,Partially ordered topological spaces , An Acad Brasil .Cienc.51(1) (1979) , 53-63 .

[3] Conder, M. S. Marshall and A. M. Slinko, Orders on multisets and discrete cones, J. Theory Ord. Sets Appl., Vol. 24, (2007), 277-296.

[4] P .Das ,Separation axioms in ordered spaces , Soochow ,J.Math.30 (4) (2004) ,447-454 .

[5] El-Sayed Abo-Tabl,Topological Rough Multisets ,WULFENIA ,vol.21,No.3 Mar 2014

[6] K.P.Girish,S.J.John,Multiset topologies induced by multiset relations ,information Sciences ,188 (2012) 298 -313 .

[7] K. P. Girish, S. J. John, Multiset topologies induced by multiset relations, Information Sciences, Vol. 188, (2012), 298-313.

[8] K. P. Girish, S. J. John, On multi topologies, Theory and Appl. Math. Comput. Sci., Vol. 2, No. 1, (2012), 37-52.

[9] K. P. Girish, S. J. John, Relations and functions in multiset context ,Information Sciences, 179 (2009) ,758 -768 .

[10] S.P.Jena ,S.K.Ghosh , B.K.Tripathy ,On the theory of bags and lists , information Sciences 132 (2001) , 241 -254 .

[11] A.Kandil , O.Tantawy , S.A.EL-Sheikh and M.Hosny ,New approach to bitopological ordered spaces , Jokull Journal 63(6) (2013) ,10-25 . Journal of Applied Mathematics ,

[12] A.Kandil , O.Tantawy , S.A.EL-Sheikh and M.Hosny ,generalization of rough sets in topological ordered spaces , Journal of Mathematical and Computational Science 4(2) (2014) ,278-297 .

[13] A.M.Kozae ,S.A.El-Sheikh and M.Hosny,On generalized rough sets closure spaces , International Journal of Applied Mathematics ,2 3 (6)(2010), 997-1023.

[14] A.M.Kozae ,S.A.El-Sheikh , M.Hosny and E.H.Aly, rough sets and its applications in a computer network , Annals of fuzzy Mathematics Informatics 6,(3) (2013), 605-624.

[15] Z.Pawlak ,Rough sets ,International- Journal of Information and Computer Sciences 11 (5) (1982), 341-356

[16] D. Singh, A. M. Ibrahim, T. Yohana and J. N. Singh. Complementation in multiset theory, Internat. math. forum, Vol. 6, No. 38, (2011), 1877-1884.

[17] D .Singh ,A .M.Ibrahim ,T.Yohanna and J.N.Singh ,An overview of the applications of Multisets ,NOV.SAD J.Math.vol.37 no. 2 , 2007 , 73 -92 .

[18] V.Venkateswaran ,A new class of multiset wilf equivalent pairs , Disvrete Math .307 ,(2007) 2508 - 2513 .

[19] R.R.Yager ,On the theory of bags , Intern.J.of general sysetems , 13 (1986) , 23 -37 .

[20] Y . Yang , X.Tan and C.Meng ,The multi-fuzzy soft set and its application in decision making , Mathematical modelling , 37 (2013) , 4915 -4923 .