Soft rough approximation operators via ideal with application in decision making

Heba.I. Mustafa

Mathematics Department, Faculty of Science,
Zagazig University, Zagazig, Egypt
dr_heba_ibrahim@yahoo.com

Abstract

Theories of soft sets and rough sets are two different approaches to vagueness. They can be combined to form a powerful mathematical tool for dealing uncertain problems. Soft rough set introduced by Feng[7] is the connection between these approaches and it is the generalization of rough set with respect to the soft approximation space. This paper extend soft rough approximation model by defining new soft rough approximation operators via ideal. When the ideal is the least ideal of $\wp(U)$, these two approximations coincide. We present the essential properties of new operators via ideal and supported by illustrative examples. The notion of soft rough equal relations via ideal is proposed and related examples are examined. We also show that rough set via ideal [26] can be viewed as a special case of soft rough set via ideal, and these two notions will coincide provided that the underlying soft set is a partition soft set. We obtain the structure of soft rough set via ideal, gives the structure of topologies induced by soft set and an ideal. Moreover, an example containing a comparative analysis between rough sets via ideal and soft rough sets via ideal is given. We show that soft rough approximation via ideal could provide a better approximation than rough set via ideal. Finally in the last section application of data reduction are done and an algorithm of multi-attribute decision making based on soft rough sets via ideal is given.

keywords: soft sets, rough approximations via ideal, soft rough sets via ideal, rough sets via ideal, soft relative postive regions via ideal, soft relative reduction via ideal.

AMS 2010 Subject Classification: 54A99

1. Introduction

In recent years vague concepts have been used in different areas as medical applications, pharmacology, economics, engineering since the classical mathematics methods are inadequate to solve many complex problems in these areas. Traditionally crisp (well-defined) property P(x) is used in mathematics, i.e., properties that are either true or false and each property defines a set: $\{x : x \textit{ has a property } P\}$[19]. Researchers have proposed many methods for vague notions. The most successful theoretical approach to the vagueness is undoubtedly fuzzy set theory [33] proposed by Zadeh in 1965. The basic idea of fuzzy set theory hinges on fuzzy membership function, which allows partial membership of elements to a set, i.e., it allows elements to belong to a set to a degree.

Rough set theory [20] is an extension of set theory for the analysis of a vague and inexact description of objects. Pawlak rough approximations are based on equivalence relation or their induced partition and subsystem, this requirement is not satisfied in many situations and thus limits the application domain of the rough set model. To solve this issue, generalizations of rough sets were considered. There are at least two approaches to generalize rough sets. One is to consider similarity, tolerance or general binary relation (see e.g.[30], [31],[32], [36]) rather than equivalence relation. The other is to extend the partition to cover (see e.g.[2, 3, 34, 36, 37]). Furthermore, as generalizations, rough sets were defined by fuzzy relation (see e.g.[5, 11, 12, 21, 22, 23, 24]) or a mapping [9, 26]. However, many of these generalizations have not been interconnected with each other.

Molodtsov [16] proposed a completely new approach, which is called soft set theory, for modelling uncertainty. Molodtsov initiated a novel concept of soft set theory [16], which is a completely new approach for modeling vagueness in 1999. A soft set is a collection of approximate descriptions of an object. Molodtsov [16, 17] presented the fundamental results of the new theory and successfully applied it to several directions such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration,theory of probability etc. He also showed that how soft set theory is free from the parametrization inadequacy syndrome of fuzzy set theory, rough set theory and etc. Fuzzy sets, rough sets and soft sets are closely related [1].

Maji et al. investigated the concept of fuzzy soft set in 2001 [13], a more generalized concept, which is a combination of fuzzy set and soft set and also studied some of its properties. This line of exploration was further investigated by several researchers [14, 28, 29]. Soft set and fuzzy soft set theories have many applications in several directions.

Feng et al. investigated the concept of soft rough set in 2010 [6] which is a combination of soft set and rough set. In [6, 7] essential properties of soft rough approximations were discussed. In [25], Shabir introduced a new approach to soft rough sets called modified soft rough set (MSR-set) and studied some of their basic properties. In [10] a new concept of soft class and soft class operations based on decision makers set were defined and some fundamental properties of soft class operations were investigated. In [18] soft rough sets and soft rough approximation operators on a complete atomic Boolean lattice were defined. Feng discussed soft set based group decision making in [8]. This study can be seen as a first attempt toward the possible application of soft rough approximations in multicriteria group decision making under vagueness.

.. It is well known that (fuzzy) ideal is an important tool for investigating rough sets (see e.g.[4, 27]). In Pawlak rough set model, any vague concept of a universe can be defined by a pair of precise concepts called the lower and upper approximations. Particularly, the empty set ϕ is a concept and the set $\{\phi\}$ is a special ideal. Hence, we have the following equivalent description of Pawlaks approximations. That is, the lower approximation contains all objects which the intersections between equivalence classes and the complement of the concept belong to $\{\phi\}$, and the upper approximation consists of all objects which the intersections between equivalence classes and the concept do not belong to $\{\phi\}$. It is a natural question to ask what does happen if we substitute a general ideal instead of the particular one. Here, the role of the ideal is to bring together some knowable and interrelated concepts of the universe, through which we can approximately obtain the imprecise concept. Since a given ideal has more concepts than that of $\{\phi\}$, the approximations based on ideals seem to enrich the Pawlaks approximations. In [27] we define new approximation operators in more general setting of complete atomic Boolean lattice by using an ideal.

The aim of this paper is to define new soft rough approximation operators in terms of an ideal. Our approach can be viewed as a generalization of several approaches that can be found in the literature. The reminder of this paper is organized as follows. In the following section, we recall some fundamental notions and propositions to be used in the present paper. In Section 3, the definition of soft rough approximations via ideal is proposed and basic properties are examined. These decrease the soft lower approximation and increase the soft upper approximation and hence increase the accuracy measure. We show by example that soft rough approximation via ideal reduce the soft boundary in comparison of soft rough approximation and the accuracy measure is better than the soft accuracy measure. So soft rough approximation via ideal could provide a better approximation than soft rough set. We also define soft rough equal relations in terms of soft rough approximation via ideal and explore some related properties. Finally, through an example we present a comparative analysis between rough set via ideal and soft rough set via ideal. In section 4 we investigate the relationships between soft sets, topologies and an ideal, obtain the structure of topologies induced by a soft set and an ideal. In section 5 we investigate the relation between soft rough via ideal and rough set via ideal [27]. We show that rough set via ideal may be considered as a special case of soft rough set via ideal. Also, we define a new pair of soft rough

approximation operators via ideal and giving the relationship between this pair and previous one.
Soft rough set approximation via ideal is a worth considering alternative to the soft rough set approximation and rogh approximation via ideal.

2. Preliminaries

In this section, we present the basic definitions and results of soft set theory which may found in earlier studies [15, 16, 17]. Throughout this paper, U refers to an initial universe, the complement of X in U is denoted by X', E is a set of parameters, $\wp(U)$ is the power set of X, and $A \subseteq E$.

Definition 2.1 [16] Let U be a universal set and E be a set of parameters. Let A be a non empty subset of E. A soft set over A, with support A ,denoted by f_A on U is defined by the set of ordered pairs

$$f_A = \{(e, f_A(e)) : e \in E, f_A(e) \in \wp(U)\},$$

or is a function $f_A : E \to \wp(U)$ s.t

$$f_A(e) \neq \phi \quad \forall \quad e \in A \subseteq E \; and \; f_A(e) = \phi \; if \; e \notin A.$$

.

From now on, we will use $S(U, E)$ instead of all soft sets over U.

Definition 2.2 [16] The soft set $f_\phi \in S(U, E)$ is called null soft set, denoted by Φ, Here $F_\phi(e) = \phi, \forall\, e \in E$.

Definition 2.3 [15] Let $f_A \in S(U, E)$. If $f_A(e) = X, \forall\, e \in A$, then f_A is called A-absolute soft set, denoted by $\widetilde{A}$.
If $A = E$, then the A-absolute soft set is called absolute soft set denoted by $\widetilde{E}_U$.

Definition 2.4 [15] Let $f_A, g_B \in S(U, E)$. f_A is a soft subset of g_B, denoted $f_A \sqsubseteq g_B$ if $f_A(e) \subseteq g_B(e), \forall\, e \in E$.

Definition 2.5 [15] Let $f_A, g_B \in S(U, E)$. Union of f_A and g_B, is a soft set h_C defined by $h_C(e) = f_A(e) \bigcup g_B(e), \forall\, e \in E$, where $C = A \cup B$. That is,

$$h_C = f_A \sqcup g_B$$

Definition 2.6 [15] Let $f_A, g_B \in S(U, E)$. Intersection of f_A and g_B, is a soft set h_C defined by $h_C(e) = f_A(e) \bigcap g_B(e), \forall\, e \in E$ where $C = A \times B$. That is

$$h_C = f_A \sqcap g_B.$$

Definition 2.7 [15] Let $f_A \in S(U, E)$. The complement of f_A, denoted by f'_A is defined by $f'_A(e) = (f(e))', \forall\, e \in E$.

Definition 2.8 [7] Let $f_A \in S(U, E)$.

i) f_A is full, if $\bigcup_{a \in A} f(a) = U$;

ii) f_A is called bijective if f_A is called full and for $a_1, a_2 \in A$ and $a_1 \neq a_2$, $f(a_1) \cap f(a_2) = \phi$

iv) f_A is called partition if $\{f(a) : a \in A\}$ forms a partition of U.

Obviously, every partition soft set is full.

Definition 2.9 [35] Let $f_A \in S(U, E)$.

i) f_A is called keeping intersection, if for any $a, b \in A$, there exists $c \in A$ such that $f(a) \cap f(b) = f(c)$;

ii) f_A is called keeping union, if for any $a, b \in A$, there exists $c \in A$ such that $f(a) \vee f(b) = f(c)$;

ii) f_A is called topological, if $\{f(a) : a \in A\}$ forms a topology on U.

Definition 2.10 [7]Let $f_A \in S(U, E)$. Then the pair $P = (U, f_A)$ is called soft approximation space. We define a pair of operators $\underline{apr}_P, \overline{apr}_P : \wp(U) \to \wp(U)$ as follows:

$\underline{apr}_P(X) = \{u \in U : \exists a \in A,\ s.t\ u \in f(a) \subseteq X\},$

$\overline{apr}_P(X) = \{u \in U : \exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X \neq \emptyset\}$

The elements $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ are called the **soft P-lower** and the **soft P-upper** approximations of X. Moreover, the sets

$Pos_P(X) = \underline{apr}_P(X)$

$Neg_P(X) = (\overline{apr}_P(X))'$

$Bnd_P(X) = \overline{apr}_P(X) - \underline{apr}_P(X)$

are called the soft P-positive region, the soft P-negative region and the soft P-boundary region of X, respectively. If $\underline{apr}_p(X) = \overline{apr}_P(X)$, X is said to be soft P-definable; otherwise X is called a soft P-rough set.

Definition 2.11[26] Let $\mathbf{B} = (B, \leq)$ be a bounded distributive lattice. A non empty subset I of B is called an ideal of B if for all $x,\ y \in B$

(i) $x,\ y \in I$ imply $x \vee y \in I$;

(ii) If $x \in I$ with $y \leq x$, then $y \in B$.

Definition 2.12[26] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \to B$ be any mapping. Let I be any ideal on B. For any element $x \in B$, let

$x^{\nabla I} = \bigvee \{x \wedge a : a \in A(B), \varphi(a) \wedge x' \in I\ \ and\ \ a \neq 0\}$, and

$x^{\triangle I} = \bigvee \{x \vee a : a \in A(B), \varphi(a) \wedge x \notin I\ \ and\ \ a \neq 1\}$.

The elements $x^{\nabla I}$ and $x^{\triangle I}$ are called the **lower** and the **upper** approximations of x via ideal I with respect to φ respectively. Two elements x and y are called equivalent via ideal I if they have the same upper and lower approximations via ideal I. The resulting equivalence classes are called rough sets via ideal I.

Proposition 2.13[26] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \to B$ be any mapping. Let I be any ideal on B, then for all $a \in A(B)$ and $x \in B$,

i) $a \leq x^{\nabla I} \Longleftrightarrow \varphi(a) \wedge x' \in I$ and $a \leq x$;

ii) $a \leq x^{\triangle I} \Longleftrightarrow \varphi(a) \wedge x \notin I$ or $a \leq x$.

Proposition 2.14 [26] Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \to B$ be any mapping. Let I be any ideal on B, then

i) $0^{\triangle I}$=0 and $1^{\nabla I}$=1;

Table 1: Tabular representation of the soft set F_A

	u_1	u_2	u_3	u_4	u_5	u_6
e_1	0	1	1	0	0	0
e_2	0	0	0	0	1	0
e_3	1	0	0	1	0	0
e_4	0	1	0	0	0	1

ii) $x \leq y$ implies $x^{\nabla I} \leq y^{\nabla I}$ and $x^{\triangle I} \leq y^{\triangle I}$.

Remark 2.15[26](1) In general, $x^{\nabla I} \leq x \leq x^{\triangle I}$.
(2) The two operations suggested in Definition 2.12 are suitable also for other operators based on binary relations. If U is any universal set, then $\wp(U)$ is a complete atomic boolean lattice whose atoms are singleton subsets of U. Let R and be a general relation on U and I any ideal on U. We define a mapping $\varphi : A(B) \longrightarrow B : U \longrightarrow \wp(U),\ x \longrightarrow R(x)$ where $R(x) = \{y \in U : xRy\}$. Then for any $X \subseteq U$, $X^{\nabla I} = \cup\{x \in U : R(x) \cap X' \in I\} \cap X$ and $X^{\triangle I} = \cup\{x \in U : R(x) \cap X \notin I\} \cup X$
If $X^{\nabla I} = X^{\triangle I}$, X is said to be R-I-definable; otherwise X is called R-I-rough set.

3. Soft Rough Approximation operators via ideal

In this section we introduce soft rough approximations via ideal and soft rough set via ideal.

Definition 3.1 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. The triple (U, f_A, I) is called soft approximation space via ideal. We define a pair of operators $\underline{apr}_I, \overline{apr}_I : \wp(U) \rightarrow \wp(U)$ as follows:

$$\underline{apr}_I(X) = \{u \in X : \exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X' \in I\},$$

$$\overline{apr}_I(X) = \{u \in U : \exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X \notin I\}$$

The elements $\underline{apr}_I(X)$ and $\overline{apr}_I(X)$ are called the **soft I-lower** and the **soft I-upper** approximations of X via ideal. In general, we refer to $\underline{apr}_I(X)$ and $\overline{apr}_I(X)$ as soft rough approximations of X with respect to P via ideal. Moreover, the sets

$$Pos_I(X) = \underline{apr}_I(X)$$

$$Neg_I(X) = (\overline{apr}_I(X))'$$

$$Bnd_I(X) = \overline{apr}_I(X) - \underline{apr}_I(X)$$

are called the soft I-positive region, the soft I-negative region and the soft I-boundary region of X, respectively. If $\underline{apr}_I(X) = \overline{apr}_I(X)$, X is said to be soft I-definable; otherwise X is called a soft I-rough set.

Proposition 3.2 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$.

Proof: Let $u \in \underline{apr}_I(X)$, then $\exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X' \in I$. If $f(a) \cap X \in I$. So, $(f(a) \cap X) \cup (f(a) \cap X') \in I$ by properties of ideal. Thus $f(a) \cap (X \cup X') = f(a) \cap U = f(a) \in I$ a contradiction. Hence $f(a) \cap X \notin I$ and consequently $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$.

By Definition 3.1, we immediately have that $X \subseteq U$ is soft I-definable if the soft I-boundary region $Bnd_I(X)$ of X is empty. Also, By Proposition 3.2, we have $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$ for all $X \subseteq U$. Nevertheless, it is worth noticing that $X \subseteq \overline{apr}_I(X)$ does not hold in general.

Example 3.3 Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $A = \{e_1, e_2, e_3, e_4\} \subseteq E$.

Table 2: Tabular representation of the soft set F_A

	u_1	u_2	u_3	u_4	u_5	u_6
e_1	1	0	0	0	0	1
e_2	0	0	0	0	1	0
e_3	0	0	0	1	0	0
e_4	1	1	0	0	1	0

Let f_A be a soft over U given by Table 1. Let I be an ideal on U defined as follows
$I = \{\phi, \{u_1\}, \{u_3\}, \{u_6\}, \{u_1, u_3\}, \{u_1, u_6\}, \{u_3, u_6\}, \{u_1, u_3, u_6\}\}$. Let $X = \{u_3, u_4, u_5\} \subseteq U$. So $X' = \{u_1, u_2, u_6\}$. Thus we have $\underline{apr}_I(X) = \{u_4, u_5\}$, and $\overline{apr}_I(X) = \{u_1, u_4, u_5\}$. So $\underline{apr}_I(X) \neq \overline{apr}_I(X)$ and X is soft I-rough set. In this case $X = \{u_3, u_4, u_5\} \not\subseteq \overline{apr}_I(X)$. Moreover, it is easy to see that $Pos_I(X) = \{u_4, u_5\}$, $Neg_I(X) = \{u_2, u_3, u_6\}$ and $Bnd_I(X) = \{u_1\}$. On the other hand, one can consider $X_1 = \{u_1, u_4, u_6\} \subseteq U$. Since $\underline{apr}_I(X_1) = \{u_1, u_4\} = \overline{apr}_I(X_1)$, then X_1 is a soft I-definable set.

Proposition 3.4 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I$ $\forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then for all $X \subseteq U$

i) $\underline{apr}_I(X) = X \cap \bigcup\{f(a) : a \in A \; and \; f(a) \cap X' \in I\}$;

ii) $\overline{apr}_I(X) = \bigcup\{f(a) : a \in A \; and \; f(a) \cap X \notin I\}$.

Proof: i) Let $u \in \underline{apr}_I(X)$. So $u \in X$ and $\exists a \in A$, $s.t \; u \in f(a)$, $f(a) \cap X' \in I$. Hence $x \in X \cap \bigcup\{f(a) : a \in A \; and \; f(a) \cap X' \in I\}$. The other inclusion can be proved similarly.

Definition 3.5 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I$ $\forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. For any $X \subseteq U$ measure of accuracy for soft set with respect to X denoted by $A_P(X)$ is defined by
$A_P(X) = \frac{|\underline{apr}_P(X)|}{|\overline{apr}_P(X)|}$
where $|\underline{apr}_P(X)|$ and $|\overline{apr}_P(X)|$, denotes the cardinalities of the sets $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ respectively. Also, measure of accuracy for soft set with respect to X via ideal denoted by $A_I(X)$ is defined by
$A_I(X) = \frac{|\underline{apr}_I(X)|}{|\overline{apr}_I(X)|}$
where $|\underline{apr}_I(X)|$ and $|\overline{apr}_I(X)|$, denotes the cardinalities of the sets $\underline{apr}_I(X)$ and $\overline{apr}_I(X)$ respectively

Now, we show in the next example that soft rough approximations via ideal provide a better approximation than soft rough approximations which provide a better approximation than rough approximations.

Example 3.6 Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $A = \{e_1, e_2, e_3, e_4\} \subseteq E$. Let f_A be a soft over U given by Table 2. Let I be an ideal on U defined as follows
$I = \{\phi, \{u_1\}, \{u_2\}, \{u_3\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_2, u_3\}, \{u_1, u_2, u_3\}\}$. Let $X = \{u_1, u_5\} \subseteq U$. So $X' = \{u_2, u_3, u_4, u_6\}$. Thus $\underline{apr}_P(X) = \{u_5\}$, $\underline{apr}_I(X) = \{u_1, u_5\} \cap \{u_1, u_2, u_5\} = \{u_1, u_5\}$, $\overline{apr}_P(X) = \{u_1, u_2, u_5, u_6\}$ and $\overline{apr}_I(X) = \{u_1, u_2, u_5\}$. So $\underline{apr}_P(X) \subseteq \underline{apr}_I(X) \subseteq X \subseteq \overline{apr}_I(X) \subseteq \overline{apr}_P(X)$. Therefore $A_P(X) = \frac{\underline{apr}_P(X)}{\overline{apr}_P(X)} = \frac{1}{4}$ and $A_I(X) = \frac{\underline{apr}_I(X)}{\overline{apr}_I(X)} = \frac{2}{3}$. Consequently, $A_I(X) > A_P(X)$. Consequently accuracy measure via ideal is better than accuracy measure for soft sets.

Proposition 3.7 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I$ $\forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal.

i) $\underline{apr}_I(\phi) = \phi = \overline{apr}_I(\phi)$

ii) $\underline{apr}_I(U) = \overline{apr}_I(U) = \bigcup_{a \in A} f(a)$;

iii) $X \subseteq Y$ implies $\underline{apr}_I(X) \subseteq \underline{apr}_I(Y)$ and $\overline{apr}_I(X) \subseteq \overline{apr}_I(Y)$.

iv) $I \subseteq J$ implies $\underline{apr}_I(X) \subseteq \underline{apr}_J(X)$

Proof: (i)Clearly, $\underline{apr}_I(\phi) = \phi$. Also, $\overline{apr}_I(\phi) = \bigcup\{f(a) : a \in A \ and \ f(a) \cap \phi \notin I\} = \bigcup\{f(a) : a \in A \ and \ \phi \notin I\} = \phi$.
(ii) $\underline{apr}_I(U) = \bigcup\{f(a) : a \in A \ and \ f(a) \cap \phi \in I\} = \bigcup\{f(a) : a \in A \ and \ \phi \in I\} = \bigcup_{a\in A} f(a)$. Also, since $f(a) \notin I \ \forall a \in A$, then $\overline{apr}_I(U) = \bigcup_{a\in A} f(a)$
(iii) Assume that $X \subseteq Y$ and $u \in \underline{apr}_I(X)$. So $u \in X$ and $\exists a \in A,\ s.t\ u \in f(a),\ f(a) \cap X' \in I$. Since $Y' \subseteq X'$, then $f(a) \cap Y' \in I$ by properties of ideal. Consequently, $u \in \underline{apr}_I(Y)$. The other part can be proved similarly.
(iv) Obvious

Proposition 3.8 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then for all $X, Y \subseteq U$

i) $\underline{apr}_I(X \cup Y) \supseteq \underline{apr}_I(X) \cup \underline{apr}_I(Y)$

ii) $\underline{apr}_I(X \cap Y) \subseteq \underline{apr}_I(X) \cap \underline{apr}_I(Y)$

iii) If f_A is keeping intersections, then $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y)$

iv) If f_A is partition, then $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y)$

v) $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) \cup \overline{apr}_I(Y)$

vi) $\overline{apr}_I(X \cap Y) \subseteq \overline{apr}_I(X) \cup \overline{apr}_I(Y)$

Proof: (i) and (ii) follow immediately by Proposition 3.7.
(iii) By (i), $\underline{apr}_I(X \cap Y) \subseteq \underline{apr}_I(X) \cap \underline{apr}_I(Y)$. Let $u \in \underline{apr}_I(X) \cap \underline{apr}_I(Y)$, then $u \in X \cap Y$ and there exists $a, b \in A$ such that $u \in f(a),\ f(a) \cap X' \in I,\ u \in f(b),\ and\ f(b) \cap X' \in I$. Since f_A is keeping intersections, then there exists $c \in A$, such that $f(a) \cap f(b) = f(c)$. By properties of ideal, $f(a) \cap f(b) \cap X' \in I$. So we prove that there exists $c \in A$, such that $u \in f(c)$ and $f(c) \cap X' \in I$. Hence $u \in \underline{apr}_I(X \cap Y)$ and consequently, $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y)$.

(iv) Let $u \in \underline{apr}_I(X) \cap \underline{apr}_I(Y)$, then $u \in X \cap Y$ and there exists $a, b \in A$ such that $u \in f(a),\ f(a) \cap X' \in I,\ u \in f(b),\ and\ f(b) \cap X' \in I$. Since f_A is partition, then $f(a) = f(b)$. So, Therefore $u \in \underline{apr}_I(X \cap Y)$. Consequently, $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y)$.

(v)By Proposition 3.7, $\overline{apr}_I(X \cup Y) \supseteq \overline{apr}_I(X) \cup \overline{apr}_I(Y)$. On the other hand, let $u \in \overline{apr}_I(X \cup Y)$, then there exists $a \in A$ such that $u \in f(a),\ f(a) \cap (X \cup Y) = (f(a) \cap X) \cup (f(a) \cap Y) \notin I$. Hence either $f(a) \cap X \notin I$ or $f(a) \cap Y \notin I$ by properties of ideal. So $u \in \overline{apr}_I(X) \cup \overline{apr}_I(Y)$ and consequently, $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) \cup \overline{apr}_I(Y)$.

(vi) Follows immediately by Proposition 3.7.

Proposition 3.9 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then for all $X \subseteq U$

i) $\overline{apr}_I(X) = \underline{apr}_I(\overline{apr}_I(X))$

ii) $\underline{apr}_I(X) \subseteq \overline{apr}_I(\underline{apr}_I(X))$

iii) $\underline{apr}_I(X) = \underline{apr}_I(\underline{apr}_I(X))$

iv) $\overline{apr}_I(X) \subseteq \overline{apr}_I(\overline{apr}_I(X))$

Proof:(i) Let $Y = \overline{apr}_I(X)$ and $u \in Y$. Then $u \in f(a)$ and $f(a) \cap X \notin I$ for some $a \in A$. By Proposition 3.4(ii), $Y = \overline{apr}_I(X) = \bigcup\{f(a) : a \in A \ and \ f(a) \cap X \notin I\}$. So there exists $a \in A$ such that $u \in f(a) \subseteq Y$. Hence $f(a) \cap Y' = \phi \in I$ and consequently, $u \in \underline{apr}_I(Y)$. Therefore $Y \subseteq \underline{apr}_I(Y)$. On the other hand, since $\underline{apr}_I(Y) \subseteq Y$ for any $Y \subseteq U$, then $Y = \underline{apr}_I(Y)$ as required.

(ii)Let $Y = \underline{apr}_I(X)$ and $u \in Y$. Then $u \in f(a)$ and $f(a) \cap X' \in I$ for some $a \in A$. But $Y =$

$\underline{apr}_I(X) = X \cap \bigcup\{f(a) : a \in A \; and \; f(a) \cap X' \in I\}$. We deduce that $u \in f(a)$ and $f(a) \cap Y = f(a) \cap X \cap \bigcup\{f(a) : a \in A \; and \; f(a) \cap X' \in I\} = f(a) \cap X$. If $f(a) \cap X \in I$, then $(f(a) \cap X) \cup (f(a) \cap X') \in I$ (by properties of ideal) i.e $f(a) \cap (X \cup' X) = f(a) \cap U = f(a) \in I$ a contradiction. Therefore, $f(a) \cap X = f(a) \cap Y \notin I$. Hence $u \in \overline{apr}_I(Y)$ and so $Y \subseteq \overline{apr}_I(Y)$.

(iii) Let $Y = \underline{apr}_I(X)$ and $u \in Y$. Then $u \in f(a)$ and $f(a) \cap X' \in I$ for some $a \in A$. But $Y = \underline{apr}_I(X) = X \cap \bigcup\{f(a) : a \in A \; and \; f(a) \cap X' \in I\}$. We deduce that $f(a) \cap X \subseteq Y$. Hence $f(a) \cap X \cap Y' = (f(a) \cap Y') \cap X = \phi$. Hence $f(a) \cap Y' \subseteq X'$ and thus $f(a) \cap Y' \subseteq f(a) \cap X'$. Since $f(a) \cap X' \in I$, then $f(a) \cap Y' \in I$. Consequently, $u \in \underline{apr}_I(Y)$. So $Y \subseteq \underline{apr}_I(Y)$.

(iv) Let $Y = \overline{apr}_I(X)$ and $u \in Y$. Then $u \in f(a)$ and $f(a) \cap X \notin I$ for some $a \in A$. But $Y = \overline{apr}_I(X) = \bigcup\{f(a) : a \in A \; and \; f(a) \cap X \notin I\}$. It follows that $u \in f(a)$ and $f(a) \cap Y = f(a) \supseteq f(a) \cap X \notin I$ by properties of ideal. So $u \in \overline{apr}_I(Y)$ and hence $Y \subseteq \overline{apr}_I(Y)$.

In the following example we indicate that the inclusion in Proposition 3.9 may be strict.

Example 3.10 Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $A = \{e_1, e_2, e_3, e_4\} \subseteq E$. Let F_A be a soft over U given by Table 2. Let I be an ideal on U defined as follows
$I = \{\phi, \{u_2\}, \{u_3\}, \{u_2, u_3\}\}$.Let $X = \{u_1, u_5, u_6\} \subseteq U$. So we have $X' = \{u_2, u_3, u_4\}$, and hence $\underline{apr}_I(X) = X \cap \{u_1, u_2, u_5, u_6\} = \{u_1, u_5, u_6\} = \{u_1, u_5, u_6\}$ and $\overline{apr}_I(X) = \{u_1, u_2, u_5, u_6\} = f(e_1) \cup f(e_2) \cup f(e_4)$. Let $Y = \overline{apr}_I(X)$. Then we have

$$\underline{apr}_I(\overline{apr}_I(X)) = \underline{apr}_I(Y) = f(e_1) \cup f(e_2) \cup f(e_4) = \overline{apr}_I(X) = Y.$$

Also, we have $\overline{apr}_I(\underline{apr}_I(X)) = \overline{apr}_I(X) = Y \underset{\neq}{\supset} X = \underline{apr}_I(X)$, which suggests that the inclusion (ii) in Proposition may hold strictly. Moreover, it is easy to see that $\underline{apr}_I(\underline{apr}_I(X)) = \underline{apr}_I(X)$.
Let $X_1 = \{u_4, u_6\}$, then $\overline{apr}_I(X_1) = \{u_1, u_4, u_6\}$. If $Y = \overline{apr}_I(X_1)$, then

$$\overline{apr}_{I_1}(\overline{apr}_I(X_1)) = \overline{apr}_I(Y_1) = \{u_1, u_2, u_4, u_5, u_6\} \underset{\neq}{\supset} Y_1 = \overline{apr}_I(X_1)$$

Proposition 3.11 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then the following properties hold

i) If f_A is keeping union, then

- **a)** for any $X \subseteq U$, there exists $a \in A$ such that $\underline{apr}_I(X) = f(a) \cap X$
- **a)** for any $X \subseteq U$, there exists $a \in A$ such that $\overline{apr}_I(X) = f(a)$

ii) If f_A is full and keeping union, then

$$\overline{apr}_I(X) = U \text{ for any } X \subseteq U \text{ such that } X \notin I$$

Proof:i) This holds by Proposition 3.4.
ii) Since f_A is full and keeping union, then $U = \bigcup_{a \in A} f(a) = f(a^*)$ for some $a^* \in A$. For each $X \subseteq U$ such that $X \notin I$ and each $u \in U$, $u \in f(a^*)$ and $f(a^*) \cap X = X \notin I$. Therefore $\overline{apr}_I(X) = U$.

Proposition 3.12 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then for any $X \subseteq U$, X is soft I-definable if and only if $\overline{apr}_I(X) \subseteq X$.

Proof: If X is soft I-definable, then $\overline{apr}_I(X) = \underline{apr}_I(X) \subseteq X$. Conversely, suppose that $\overline{apr}_I(X) \subseteq X$ for $X \subseteq U$. Since $f(a) \notin I \; \forall a \in A$, then $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$ by Proposition 3.2. To show that X is soft I-definable, it remains to prove that $\overline{apr}_I(X) \subseteq \underline{apr}_I(X)$. Let $u \in \overline{apr}_I(X)$. Then $\exists a \in A$, $s.t \; u \in f(a)$, $f(a) \cap X \notin I$. It follows that $u \in f(a) \subseteq \overline{apr}_I(X) \subseteq X$. So $u \in X$, $u \in f(a)$ and $f(a) \cap X' = \phi \in I$. Therefore $u \in \underline{apr}_I(X)$ and so $\overline{apr}_I(X) \subseteq \underline{apr}_I(X)$ as required.

Example 3.13 To illustrate the above result, we revisit Example 3.6. Let $X = \{u_2, u_4\} \subseteq U$. So $X' = \{u_1, u_3, u_5, u_6\}$, $\underline{apr}_I(X) = \{u_4\} = \overline{apr}_I(X)$. Hence $\overline{apr}_I(X) \subseteq X$ and X is soft I-definable set. On the other hand, for $X_1 = \{u_4, u_6\} \subseteq U$, $X_1{}' = \{u_1, u_2, u_3, u_5\}$, $\underline{apr}_I(X_1) = \{u_4, u_6\} \cap \{u_1, u_4, u_6\} = \{u_4, u_6\}$ and $\overline{apr}_I(X_1) = \{u_1, u_4, u_6\}$. Thus $\overline{apr}_I(X_1) \nsubseteq X$ and X_1 is soft I-rough set.

Proposition 3.14 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. The following conditions are equivalent

i) S is a full soft set.

ii) $\underline{apr}_I(U) = U$

iii) $\overline{apr}_I(U) = U$

Proof: $\underline{apr}_I(U) = U \cap (\bigcup\{f(a) : a \in A\ and\ f(a) \cap U' \in I\}) = \bigcup\{f(a) : a \in A\ and\ f(a) \cap \phi \in I\}) = \bigcup\{f(a) : a \in A\ and\ \phi \in I\}) = \bigcup_{a \in A} f(a)$.
Hence by definition, $S = (f, A)$ is a full soft set if and only if $\underline{apr}_I(U) = U$. That is, conditions (i) and (ii) are equivalent. Similarly, we can show that (i) and (iii) are equivalent conditions.

Proposition 3.15 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. The following conditions are equivalent

i) $X \subseteq \overline{apr}_I(X)\ \forall\ X \subseteq U$

ii) $\overline{apr}_I(\{u\}) \neq \phi\ \forall\ u \in U$

Proof: Assume that (i) holds, then $\{u\} \subseteq \overline{apr}_I(\{u\})\ \forall\ u \in U$i.e, $\overline{apr}_I(\{u\}) \neq \phi$.
Assume that (ii) holds. Let $u \in X$, so by (ii)$\overline{apr}_I(\{u\}) \neq \phi$. Let $v \in \overline{apr}_I(\{u\})$, then $\exists a \in A,\ s.t\ v \in f(a) and\ f(a) \cap \{u\} \notin I$. So $f(a) \cap \{u\} \neq \phi$. It follows that $u = v \in f(a)$. Since $f(a) \cap \{u\} \notin I$ and $\{u\} \subseteq X$, then $f(a) \cap X \notin I$. Consequently, $u \in \overline{apr}_I(X)$.

Proposition 3.16 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. If $\overline{apr}_I(\{u\}) \neq \phi\ \forall\ u \in U$, then for any $X \subseteq U$

i) $(\underline{apr}_I(X))' \subseteq \overline{apr}_I(X')$

ii) $Neg_I(X) = (\overline{apr}_I(X))' \subseteq \underline{apr}_I(X')$

Proof: If $(\underline{apr}_I(X))'$ is empty, then clearly we have the inclusion (i). Suppose $(\underline{apr}_I(X))' \neq \phi$. Let $u \in (\underline{apr}_I(X))'$. Since f_A is full, then $\exists a_o \in A,\ s.t\ u \in f(a_o)$. Note also that
$(\underline{apr}_I(X))' = \{u \in U : \forall a \in A,\ u \in f(a) \Rightarrow f(a) \cap X' \notin I\} \cup X'$. Thus it follows that either $u \in X'$ or $f(a_o) \cap X' \notin I$ since $u \in f(a_o)$. If $u \in X'$, since $\overline{apr}_I(\{u\}) \neq \phi\ \forall\ u \in U$, then $X' \subseteq \underline{apr}_I(X')$ by Proposition 3.15. Therefore $u \in \overline{apr}_I(X')$. If $f(a_o) \cap X' \notin I$, then $u \in \overline{apr}_I(X')$. Consequently, $(\underline{apr}_I(X))' \subseteq \overline{apr}_I(X')$.
(ii)It is clear that the inclusion $Neg_I(X) = (\overline{apr}_I(X))' \subseteq \underline{apr}_I(X')$ holds when the set $(\overline{apr}_I(X))'$ is empty. So suppose that $(\overline{apr}_I(X))' \neq \phi$. Let $u \in (\overline{apr}_I(X))'$. Since $\overline{apr}_I(\{u\}) \neq \phi\ \forall\ u \in U$, then $X \subseteq \overline{apr}_I(X)$ by Proposition 3.15 and thus $u \in X'$. Since f_A is full, then $\exists a_o \in A,\ s.t\ u \in f(a_o)$. But we have that
$Neg_I(X) = (\overline{apr}_I(X))' = \{u \in U : \forall a \in A,\ u \in f(a) \Rightarrow f(a) \cap X \in I\}$. Thus it follows that $f(a_o) \cap (X')' \in I$ since $u \in f(a_o)$. Therefore $u \in \underline{apr}_I(X')$.

Definition 3.17 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Let $X \subseteq U$, We define the following seven types of soft rough sets via ideal

i) X is roughly soft I-definable if $\underline{apr}_I(X) \neq \phi$ and $\overline{apr}_I(X) \neq U$

ii) X is internally soft I-definable if $\underline{apr}_I(X) = \phi$ and $\overline{apr}_I(X) \neq U$

iii) X is externally soft I-definable if $\underline{apr}_I(X) \neq \phi$ and $\overline{apr}_I(X) = U$

iv) X is totally soft I-definable if $\underline{apr}_I(X) = \phi$ and $\overline{apr}_I(X) = U$

iv) X is externally soft P-I-definable if $\underline{apr}_I(X) \neq \phi$ and $\overline{apr}_P(X) = U$

iv) X is internally soft P-I-definable if $\underline{apr}_P(X) = \phi$ and $\overline{apr}_I(X) \neq U$

Proposition 3.18 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I$ $\forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Let $X \subseteq U$.

i) If X is roughly soft P-definable, then it is roughly soft I-definable.

ii) If X is totally soft I-definable, then it is totally soft P-definable.

Proof: Obvious.

Definition 3.19 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I$ $\forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. For any $X, Y \subseteq U$ we define

i) $X \sim_I Y \Longleftrightarrow \underline{apr}_I(X) = \underline{apr}_I(Y)$

ii) $X \sim^I Y \Longleftrightarrow \overline{apr}_I(X) = \overline{apr}_I(Y)$

iii) $X \approx_I Y \Longleftrightarrow X \sim_I Y$ and $X \sim^I Y$

These binary relations are called the lower soft rough equal relation via ideal,the upper soft rough equal relation via ideal, and the soft rough equal relation via idea, respectively.
It is easy to verify that the relations defined above are all equivalence relations over $\wp(U)$.

Proposition 3.20 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I$ $\forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. For any $X, Y \subseteq U$ we have

i) $X \sim^I Y \Longleftrightarrow X \sim^I (X \cup Y) \sim^I Y$

ii) $X \sim^I X_1$,$Y \sim^I Y_1 \Longrightarrow (X \cup Y) \sim^I (X_1 \cup Y_1)$

iii) $X \sim^I Y \Longrightarrow X \cup (Y') \sim^I U$

iv) $X \subseteq Y$, $Y \sim^I \phi \Longleftrightarrow X \sim^I \phi$

v) $X \subseteq Y$, $X \sim^I U \Longleftrightarrow Y \sim^I U$

Proof:(i)If $X \sim^I Y$, then $\overline{apr}_I(X) = \overline{apr}_I(Y)$. Since $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) \cup \overline{apr}_I(Y)$, we deduce $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) = \overline{apr}_I(Y)$ and so $X \sim^I (X \cup Y) \sim^I Y$. Conversely, if $X \sim^I (X \cup Y) \sim^I Y$, then we immediately have that $X \sim^I Y$ by using the transitivity of the relation $\sim^I$.
(ii) Assume that $X \sim^I X_1$ and $Y \sim^I Y_1$. Then by definition, we know that $\overline{apr}_I(X) = \overline{apr}_I(X_1)$ and $\overline{apr}_I(Y) = \overline{apr}_I(Y_1)$. Since $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X) \cup \overline{apr}_I(Y)$ and $\overline{apr}_I(X_1 \cup Y_1) = \overline{apr}_I(X_1) \cup \overline{apr}_I(Y_1)$, we deduce that $\overline{apr}_I(X \cup Y) = \overline{apr}_I(X_1 \cup Y_1)$, whence $(X \cup Y) \sim^I (X_1 \cup Y_1)$.
(iii) Suppose that $X \sim^I Y$. Then by definition, $\overline{apr}_I(X) = \overline{apr}_I(Y)$. Since $\overline{apr}_I(X \cup Y') = \overline{apr}_I(X) \cup \overline{apr}_I(Y')$ and $\overline{apr}_I(U) = \overline{apr}_I(Y) \cup \overline{apr}_I(Y')$, it follows that $\overline{apr}_I(X \cup Y') = \overline{apr}_I(U)$; hence $X \cup (Y') \sim^I U$ as required.
(iv) Let $X \subseteq Y$ and $Y \sim^I \phi$. Then we deduce $\overline{apr}_I(X) \subseteq \overline{apr}_I(Y) = \overline{apr}_I(\phi) = \phi$.
Hence $\overline{apr}_I(X) = \phi = \overline{apr}_I(\phi)$, and so we have that $X \sim^I \phi$.
(v) Suppose that $X \subseteq Y$ and $X \sim^I U$. Then we deduce $\overline{apr}_I(Y) \supseteq \overline{apr}_I(X) = \overline{apr}_I(U)$. Since $Y \subseteq U$, then $\overline{apr}_I(Y) \supseteq \overline{apr}_I(U)$. Therefore $\overline{apr}_I(Y) = \overline{apr}_I(U)$, and so $Y \sim^I Y$ as required.

Proposition 3.21 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I$ $\forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. If f_A is keeping intersection, then for any $X, Y \subseteq U$ we have

Table 3: An information table

	u_1	u_2	u_3	u_4	u_5	u_6
Sex	Woman	Woman	Man	Man	Man	Man
Age category	Young	Young	$Matureage$	Old	$Mature\ age$	$Baby$
Living area	City	City	$City$	$Village$	$City$	$Village$
Habits	NSND	NSND	$Smoke$	SD	$Smoke$	$NSND$

i) $X \sim_I Y \Longleftrightarrow X \sim_I (X \cap Y) \sim_I Y$

ii) $X \sim_I X_1, Y \sim_I Y_1 \Longrightarrow (X \cap Y) \sim_I (X_1 \cap Y_1)$

iii) $X \sim_I Y \Longrightarrow X \cap (Y') \sim_I \phi$

iv) $X \subseteq Y,\, Y \sim_I \phi \Longrightarrow X \sim_I \phi$

v) $X \subseteq Y,\, X \sim_I U \Longleftrightarrow Y \sim_I U$

Proposition 3.22 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then for any $X \subseteq U$

$$\underline{apr}_I(X) = \bigcap\{Y \subseteq U : X \sim_I Y\}$$

Proof: Let $u \in \underline{apr}_I(X)$. If $X \sim_I Y$, then by definition $\underline{apr}_I(X) = \underline{apr}_I(Y)$. But $\underline{apr}_I(Y) \subseteq Y$ for any $Y \subseteq U$. It follows that $u \in \underline{apr}_I(X) = \underline{apr}_I(Y) \subseteq Y$.
Hence $u \in \bigcap\{Y \subseteq U : X \sim_I Y\}$, and so $\underline{apr}_I(X) \subseteq \bigcap\{Y \subseteq U : X \sim_I Y\}$. Next, we show that the reverse inclusion $\bigcap\{Y \subseteq U : X \sim_I Y\} \subseteq \underline{apr}_I(X)$ also holds. Let $u \in \bigcap\{Y \subseteq U : X \sim_I Y\}$. Then by Proposition 3.9, we have $\underline{apr}_I(X) = \underline{apr}_I(\underline{apr}_I(X))$. Thus $X \sim_I \underline{apr}_I(X)$, and it follows that $u \in \underline{apr}_I(X)$. Consequently, we conclude that $\underline{apr}_I(X) = \bigcap\{Y \subseteq U : X \sim_I Y\}$.

Example 3.23 As in Example 3.6. Let $X = \{u_4, u_5, u_6\} \subseteq U$. So we have $X' = \{u_1, u_2, u_3\}$, and hence $\underline{apr}_I(X) = X \cap \{u_1, u_2, u_4, u_5, u_6\} = \{u_4, u_5, u_6\} = X$. It is easy to see that

$$\underline{apr}_I(X) = \bigcap\{Y \subseteq U : X \sim_I Y\}.$$

Example 3.24 Let us consider the following soft set $S = f_E$ which describes life expectancy. Suppose that the universe $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ consists of six persons and $E = \{e_1, e_2, e_3, e_4\}$ is a set of decision parameters. The e_i (i = 1,2,3,4) stands for "under stress", "young", "drug addict" and "healthy". Set $f(e_1) = \{u_1, u_6\}$, $f(e_2) = \{u_5\}$, $f(e_3) = \{u_4\}$; and $f(e_4) = \{u_1, u_2, u_6\}$. The soft set f_E can be viewed as the following collection of approximations:
$f_E = \{(understress, \{u_1, u_6\}); (young, \{u_5\}); (drugaddict, \{u_4\}); (healthy; \{u_1, u_2, u_6\})\}$.
On the other hand, "life expectancy" topic can also be described using rough sets as follows: The evaluation will be done in terms of attributes: "sex", "age category", "living area", "habits", characterized by the value sets "{man, woman}", "{baby, young, mature age, old}", "{village, city}", "{smoke, drinking, smoke and drinking, no smoke and no drinking}". We denote "smoke and drinking" by SD and "no smoke and no drinking" by NSND. The information will be given by Table 3, where the rows are labeled by attributes and the table entries are the attribute values for each person. From here we obtain the following equivalence classes, induced by the above mentioned attributes:
$[u_1]_R = [u_2]_R = \{u_1, u_2\}$, $[u_3]_R = [u_5]_R = \{u_3, u_5\}$, $[u_4]_R = \{u_4\}$, $[u_6]_R = \{u_6\}$.
Let I be an ideal on U defined as follows
$I = \{\phi, \{u_2\}, \{u_3\}, \{u_2, u_3\}\}$.
Let X be a target subset of U, that we wish to represent using the above equivalence classes. Hence we analyze the upper and lower approximations of X, in some particular cases:
1. Let $X = \{u_5\}$. It follows that
$X^{\nabla I} = \{u_5\}$, $X^{\triangle I} = \{u_3, u_5\}$. So X is R-I-rough.
Let us calculate now the soft I-lower and I-upper approximations of X. We obtain
$\underline{apr}_I(X) = \{u_5\} = X$, $\overline{apr}_I(X) = \{u_5\} = X$
hence X is soft I-definable.
2. Let $X = \{u_2, u_5\}$. It follows that $\underline{apr}_I(X) = \{u_5\} = \overline{apr}_I(X)$. So X is soft I-definable. On the other

hand, $\underline{apr}_P(X) = \{u_5\}$, $\overline{apr}_P(X) = \{u_1, u_2, u_5, u_6\}$, hence X is soft P-rough.

The above results show that soft rough set approximation via ideal is a worth considering alternative to the rough set approximation via ideal. Soft rough sets via ideal could provide a better than rough sets via ideal do, depending on the structure of the equivalence approximation classes and of the subsets f(e), where $e \in E$.

.

4. The relations among soft sets, ideal and topologies

In this section, we investigate the relationship between topological soft sets, topologies and an ideal.

Theorem 4.1 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. If f_A is full, then

i) $\tau_f = \{X \subseteq U : X = \underline{apr}_I(X)\}$ is a generalized topology on U.

ii) If f_A is keeping intersections, then τ_f is a topology on U.

Proof: Since $\underline{apr}_I(\phi) = \phi$, then $\phi \in \tau_f$. Let $\Im \subseteq \tau_f$. Denote $\Im = \{X_\alpha : \alpha \in \Gamma\}$ where Γ is an index set. Put $X = \bigcup\{X_\alpha : \alpha \in \Gamma\}$. Since $X_\alpha \subseteq X$ for each $\alpha \in \Gamma$, then $X_\alpha = \underline{apr}_I(X_\alpha) \subseteq \underline{apr}_I(X)$ by Proposition 3.7. So $X = \bigcup\{X_\alpha : \alpha \in \Gamma\} \subseteq \underline{apr}_I(X)$. Thus $\underline{apr}_I(X) = X$. This implies $\bigcup\{X_\alpha : \alpha \in \Gamma\} \in \tau_f$. Hence τ_f is a generalized topology on U.
(ii) By Propositions and $\underline{apr}_I(U) = U$ and thus $U \in \tau_f$. Let $X, Y \in \tau_f$, then $\underline{apr}_I(X \cap Y) = \underline{apr}_I(X) \cap \underline{apr}_I(Y) = X \cap Y$ by Proposition 3.8. So $X \cap Y \in \tau_f$. By (i) τ_f is a generalized topology on U. Thus τ_f is a topology on U.

Definition 4.2 Let $f_A \in S(U, E)$ be full and keeping intersections and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then τ_f is called the topology induced by f_A and an ideal I on U.

The following theorem gives the topological structure on soft sets and an ideal(i.e. the structure of topology induced by soft set and an ideal).

Theorem 4.3 Let $f_A \in S(U, E)$ be full and keeping intersections and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then

i) $\{\overline{apr}_I(X) : X \subseteq U\} \subseteq \tau_f = \{\underline{apr}_I(X) : X \subseteq U\}$

ii) $\tau_f \supseteq \{f(a) : a \in A\}$

iii) $\underline{apr}_I(X)$ is an interior operator of τ_f

Proof: (i) Since $\overline{apr}_I(X) = \underline{apr}_I(\overline{apr}_I(X))$ by Proposition 3.9, then $\{\overline{apr}_I(X) : X \subseteq U\} \subseteq \tau_f$.
Obviously,

$$\tau_f \subseteq \{\underline{apr}_I(X) : X \subseteq U\}$$

Let $Y \in \{\underline{apr}_I(X) : X \subseteq U\}$. Then $Y = \underline{apr}_I(X)$ for some $X \subseteq U$. By Proposition 3.9, $\underline{apr}_I(X) = \underline{apr}_I(\underline{apr}_I(X))$. So $Y \in \tau_f$. Thus $\tau_f \supseteq \{\underline{apr}_I(X) : X \subseteq U\}$. Hence $\{\overline{apr}_I(X) : X \subseteq U\} \subseteq \tau_f = \{\underline{apr}_I(X) : X \subseteq U\}$ as required.

(ii) For each $a \in A$, by Proposition 3.4 $\underline{apr}_I(f(a)) = f(a) \cap \bigcup\{f(a^*) : a^* \in A, f(a^*) \cap (f(a))' \in I\} \subseteq f(a)$. Since $f(a) \cap (f(a))' = \phi \in I$, then $f(a) \subseteq f(a) \cap \bigcup\{f(a^*) : a^* \in A, f(a^*) \cap (f(a))' \in I\} = \underline{apr}_I(f(a))$. Hence $f(a) = \underline{apr}_I(f(a))$ and so $f(a) \in \tau_f$. Therefore $\{f(a) : a \in A\} \subseteq \tau_f$.

(iii)It suffices to show that $\underline{apr}_I(X) = int(X) \ \forall X \subseteq U$. By (i) $\underline{apr}_I(X) \in \tau_f$ and since $\underline{apr}_I(X) \subseteq X$, then

$\underline{apr}_I(X) \subseteq int(X)$. Conversely, let $Y \in int(X)$, then $Y \in \tau_f$ and $Y \subseteq X$. So $Y = \underline{apr}_I(Y) \subseteq \underline{apr}_I(X)$. Thus $int(X) = \bigcup\{Y : Y \in \tau_f, Y \subseteq X\} \subseteq \underline{apr}_I(X)$. Consequently, $\underline{apr}_I(X) = int(X)$.

Definition 4.4 Let τ be a topology on U and I be an ideal on U. Put $\tau = \{U_a : a \in A \; and \; U_a \notin I\}$ where A is the set of indexes. Define a mapping $f_\tau : A \to \wp(U)$ by $f_\tau(a) = U_a$ for each $a \in A$. Then, the soft set $(f_\tau)_A$ over U is called the soft set induced by τ on U and an ideal I on U.

Proposition 4.5 (1)Let τ be a topology on U and I be an ideal on U. Let $(f_\tau)_A$ be the soft set induced by τ and I on U. Then, $(f_\tau)_A$ is a full, keeping intersection, keeping union soft over U and $(f_\tau)_A \notin I$ for each $a \in A$.

(2) Let τ_1 and τ_2 be two topologies on U and I_1 and I_2 be two ideals on U. Let $(f_{\tau_1})_{A_1}$ and $(f_{\tau_2})_{A_2}$ be two soft sets induced, respectively, by τ_1 and I_1 and, τ_2 and I_2 on U.
If $\tau_1 \subseteq \tau_2$, then

$$(f_{\tau_1})_{A_1} \supseteq (f_{\tau_2})_{A_2}$$

Proof: Obvious.

Proposition 4.6 Let τ be a topology on U, let I be an ideal on U such that $G \notin I \; \forall G \in \tau$. Then there exists a full, keeping intersection, and keeping union soft set f_A with $f_A(a) \notin I$ for each $a \in A$ such that $\underline{apr}_I(X) \supseteq int(X)$ for each $X \in \wp(U)$ where (U, f_A, I) be a soft approximation space via ideal.

Proof: Put $\tau = \{U_a : a \in A\}$, where A is the set of indexes. Define a mapping $f : A \to \wp(U)$ by

$$f(a) = U_a \text{ for each } a \in A$$

By Proposition 4.5 f_A is full, keeping intersection, and keeping union and $f_A(a) \notin I$ for each $a \in A$. Now, we show that $\underline{apr}_I(X) \supseteq int(X)$ for each $X \in \wp(U)$. Let $X \in \wp(U)$ and $x \in int(X)$, then $\exists$ open neighbourhood W of x s.t $W \subseteq X$. So, $W = U_a$ for some $a \in A$. This implies $x \in U_a = f(a)$ and $f(a) \cap X' = \phi \in I$. Therefore $x \in \underline{apr}_I(X)$. Consequently, $\underline{apr}_I(X) \supseteq int(X)$.

Theorem 4.7 Let f_A be full and keeping intersections soft set over U and I be an ideal on U such that $f(a) \notin I \; \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Let τ_f be the topology induced by f_A and I on U. Let $(f_{\tau_f})_B$ be the soft set induced by τ_f and I on U. Then

$$f_A \subseteq (f_{\tau_f})_B$$

Proof: By Theorem 4.3 $\tau_f \supseteq \{f(a) : a \in A\}$. Let $\tau_f = \{U_a : U_a \notin I, a \in B\}$, where $A \subseteq B$, $U_a = f(a) \; \forall \; a \in A$. Therefore $f_{\tau_f} : B \to \wp(U)$, where $f_{\tau_f}(a) = U_a$ for each $a \in B$. Hence $f_A \subseteq (f_{\tau_f})_B$.

5. The relations between soft rough approximation via ideal and rough approximation via ideal

In this section we will describe the relationship between rough sets via ideal and soft rough sets via ideal.

Definition 5.1 Let R be a binary relation on U and I be an ideal on U such that $R(a) \notin I \; \forall a \in U$. Define a mapping $f_R : U \to \wp(U)$ by

$$f_R(a) = R(a)$$

for each $a \in A$, where $A = U$. Then, $(f_R)_A$ is called the soft set induced by R and I on U.

Theorem 5.2 Let R be an equivalence relation on U and I be an ideal on U such that $R(a) \notin I$

$\forall a \in U$. Let $(f_R)_A$ be the soft set induced by R and I on U. Let $(U, (f_R)_A, I)$ be a soft approximation space via ideal. If $\overline{apr}_I(\{u\}) \neq \phi \ \forall\ u \in U$, then for all $X \subseteq U$, $X^{\nabla I} = \underline{apr}_I(X)$ and $X^{\triangle I} = \overline{apr}_I(X)$. Thus in this case,

i) $X \subseteq U$ is R-I-definable iff X is a soft I-definable set.

ii) $X \subseteq U$ is R-I-rough iff X is a soft I-rough set.

Proof: Let $X \subseteq U$ and $u \in U$. We show that $X^{\nabla I} = \underline{apr}_I(X)$. If $u \in \underline{R}_I(X) = \{x \in X : [x]_R \cap X' \in I\}$, then $[u]_R \cap X' \in I$. So, $\exists\ u \in X$ s.t $u \in [u]_R = f_R(u) \cap X' \in I$. Therefore $u \in \underline{apr}_I(X)$, and so $X^{\nabla I} \subseteq \underline{apr}_I(X)$. Conversely, assume that $u \in \underline{apr}_I(X)$. So,$u \in X$ and $\exists\ v \in U$ s.t $u \in f_R(v) = [v]_R$, $[v]_R \cap X' \in I$. It follows that $[u]_R = [v]_R$. Thus $[u]_R \cap X' = [v]_R \cap X' \in I$ and $u \in X^{\nabla I}$. Consequently, $X^{\nabla I} = \underline{apr}_I(X)$.
Now we show that $X^{\triangle I} = \overline{apr}_I(X)$. Let $u \in X^{\triangle I}$, then either $u \in X$ or $[u]_R \cap X \notin I$. If $u \in X$, then $u \in \overline{apr}_I(X)$ by Proposition 3.15 since $\overline{apr}_I(\{u\}) \neq \phi \ \forall\ u \in U$. If $[u]_R \cap X \notin I$, then $\exists\ u \in U$ s.t $u \in [u]_R = f_R(u) \cap X \notin I$ and therefore $u \in \overline{apr}_I(X)$. Therefore $X^{\triangle I} \subseteq \overline{apr}_I(X)$. Conversely, let $u \in \overline{apr}_I(X)$. Then $\exists\ v \in U$ s .t $u \in f_R(v) = [v]_R$, $[v]_R \cap X \notin I$. Thus $[u]_R = [v]_R$ and $[u]_R \cap X \notin I$. Hence $u \in X^{\triangle I}$ and consequently $X^{\triangle I} = \overline{apr}_I(X)$.

Definition 5.3 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$
(i) Define a binary relation R_f on U by

$$xR_f y \Leftrightarrow \exists\ a \in A\ , \{x, y\} \subseteq f(a)$$

for each $x, y \in U$. Then R_f is called the binary relation induced by f_A and I on U.
(ii) For each $x \in U$, define a successor neighbourhood $(R_f)_s(x) = \{y \in U : xR_f y\}$

Proposition 5.4 [35] Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let R_f be the binary relation induced by f_A on U. Then, the following properties hold.

i) R_f is a symmetric relation.

ii) If f_A is full, then R_f is a reflexive relation.

iii) If f_A is a partition, then R_f is an equivalence relation.

Proposition 5.5 [35] Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let R_f be the binary relation induced by f_A on U. Then, the following properties hold.

i) If $u \in f(a)$ for $a \in A$, then $f(a) \subseteq R_f(u)$.

ii) If f_A is a partition and $u \in f(a)$ for $a \in A$, then $f(a) = R_f(u)$.

iii) If f_A is keeping union, then for all $u \in U\ \exists a \in A,\ s.t\ R_f(u) = f(a)$.

Next, we define a new pair of soft rough approximation operators via ideal and giving the relationship between this pair and previous one.

Definition 5.6 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. We define a pair of operators $\underline{apr}'_P, \overline{apr}'_P : \wp(U) \to \wp(U)$ as follows:

$\underline{apr}'_I(X) = \{x \in X : R_f(x) \cap X' \in I\}$,

$\overline{apr}'_I(X) = \{x \in U : R_f(x) \cap X \notin I\} \bigcup X$

Proposition 5.7 Let $f_A \in S(U, E)$ be partition and I be an ideal on U such that $f(a) \notin I \ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Let R_f be a binary relation induced by f_A on U. Then, the following properties hold for any $X \subseteq U$

i) If f_A is full, then

$$\underline{apr}_I(X) \supseteq \underline{apr}'_I(X)$$

ii) If f_A is full, keeping union and $X \notin I$, then

$$\overline{apr}_I(X) \supseteq \overline{apr}'_I(X)$$

iii) If f_A is partition, then

a) $\underline{apr}_I(X) = \underline{apr}'_I(X)$

b) If $\overline{apr}_I(\{u\}) \neq \phi\ \forall\ u \in U$, then $\overline{apr}_I(X) = \overline{apr}'_I(X)$

Proof: i) Suppose that $x \in \underline{apr}'_I(X)$. Then $x \in X$ and $R_f(x) \cap X' \in I$. Since f_A is full, then $x \in f(a)$ for some $a \in A$. By Proposition 5.5 $f(a) \subseteq R_f(x)$. Thus, $x \in f(a)$ and $f(a) \cap X' \in I$ by properties of ideal. Consequently, $x \in \underline{apr}_I(X)$. So,

$$\underline{apr}_I(X) \supseteq \underline{apr}'_I(X)$$

ii)Since $X \notin I$, then $X \neq \phi$. By Proposition 3.11(ii), $\overline{apr}_I(X) = U$. Thus

$$\overline{apr}_I(X) \supseteq \overline{apr}'_I(X)$$

iii) a) Suppose that $x \in \underline{apr}_I(X)$. Then, $x \in X$ and $\exists\ a \in A$ s.t $x \in f(a)$ and $f(a) \cap X' \in I$. Since f_A is partition and $x \in f(a)$, then $f(a) = R_f(x)$ by Proposition 3.11. This implies that $x \in \underline{apr}'_I(X)$. Therefore

$$\underline{apr}_I(X) \subseteq \underline{apr}'_I(X)$$

Since every partition soft set is full, then by i)

$$\underline{apr}_I(X) = \underline{apr}'_I(X)$$

iii) b) Suppose that $x \in \overline{apr}_I(X)$. Then, $\exists\ a \in A$ s.t $x \in f(a)$ and $f(a) \cap X \notin I$. Since f_A is partition and $x \in f(a)$, then $f(a) = R_f(x)$ by Proposition 3.11. This implies that $x \in \overline{apr}'_I(X)$. Therefore

$$\overline{apr}_I(X) \subseteq \overline{apr}'_I(X)$$

Suppose that $x \in \overline{apr}'_I(X)$. Then, either $x \in X$ or $R_f(x) \cap X \notin I$. If $x \in X$, since $\overline{apr}_I(\{u\}) \neq \phi\ \forall\ u \in U$, then $X \subseteq \overline{apr}_I(X)$ by Proposition 3.15 and therefore $x \in \overline{apr}_I(X)$. If $R_f(x) \cap X \notin I$, since f_A is full, then $x \in f(a)$ for some $a \in A$. Since f_A is partition and $x \in f(a)$, then $f(a) = R_f(x)$ by Proposition 3.11. This implies that $x \in \overline{apr}_I(X)$. Therefore

$$\overline{apr}'_I(X) \subseteq \overline{apr}_I(X)$$

Hence $\overline{apr}_I(X) = \overline{apr}'_I(X)$.

Theorem 5.8 Let $f_A \in S(U, E)$ be partition and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Let R_f be a binary relation induced by f_A on U. Then, for all $X \subseteq U$, $X^{\nabla I} = \underline{apr}_I(X) = \underline{apr}'_I(X)$ and $X^{\triangle I} = \overline{apr}_I(X) = \overline{apr}'_I(X)$.
where $X^{\nabla I_f}$ and $X^{\triangle I_f}$ are the rough approximations operators of X via ideal.
Proof: Follows immediately by Propositions 5.5 and 5.7.

Remark 5.9 Theorems 5.2 and 5.8 illustrate that rough set models via ideal can be viewed as a special case of soft rough sets via ideal.

Proposition 5.10 Let $f_A \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal and R_f be a binary relation induced by f_A on U.

i) If $X \subseteq U$ is R_f-I- definable, then X is soft I-definable.

ii) If $X \subseteq U$ is R_f-I- Rough, then X is soft I-Rough.

Proof: (i) If $X = \phi$, then X is soft I-definable by Proposition 3.7. Let $\phi \neq X \in \wp(U)$ be R-I-definable. by Proposition 3.2, $\underline{apr}_I(X) \subseteq \overline{apr}_I(X)$. It remains to show that $\overline{apr}_I(X) \subseteq \underline{apr}_I(X)$. Let $u \in \overline{apr}_I(X)$, then there exists $a \in A$ such that $u \in f(a)$ and $f(a) \cap X \notin I$. By Proposition 5.5, $f(a) \subseteq R_f(u)$. Since $f(a) \cap X \notin I$, then $R_f(u) \cap X \notin I$ by Properties of ideal. But $u \in R_f(u)$, so $u \in X^{\triangle I} = X^{\nabla I}$. Hence $u \in X$ and $R_f(u) \cap X' \in I$. Therefore $f(a) \cap X' \in I$ by Properties of ideal and thus $u \in \underline{apr}_I(X)$. Consequently, $\overline{apr}_I(X) \subseteq \underline{apr}_I(X)$. So X is soft I-definable.
(ii)Follows immediately by (i).

The following example shows that the converse of the above proposition is not true in general.

Example 5.11 Let $U = \{h_1, h_2, h_3, h_4, h_5\}$. Let I be an ideal on U and let R be a binary relation on U, defined as follows:

$I = \{\{h_1\}, \{h_2\}, \{h_1, h_2\}, \phi\}$ and let f_A be a soft set over U defined as follows

$f(a_1) = \{h_1, h_4\}$, $f(a_2) = \{h_4\}$, $f(a_3) = \{h_2, h_3, h_5\}$, $f(a_4) = \{h_1, h_2, h_4\}$.
Let R be the binary relation induced by f_A. Then

$R(h_1) = \{h_1, h_2, h_4\}$, $R(h_2) = \{h_1, h_2, h_3, h_4, h_5\}$, $R(h_3) = \{h_2, h_3, h_5\}$, $R(h_4) = \{h_1, h_2, h_4\}$, $R(h_5) = \{h_2, h_3, h_5\}$.
Let $X = \{h_2, h_3, h_5\} \subseteq U$. So $X' = \{h_1, h_4\}$. Thus $X^{\nabla I} = \{h_3, h_5\}$, and $X^{\triangle I} = \{h_2, h_3, h_5\}$. Also, $\underline{apr}_I(X) = \{h_2, h_3, h_5\}$, $\overline{apr}_I(X) = \{h_2, h_3, h_5\}$.

Then X is an R_f-I-rough set. But X is soft I-definable set.

6 Application of data reduction using soft rough set via ideal

Definition 6.1 Let $f_{iC_i} \in S(U, E)$ (i=1,2,.....n)be a bijective soft sets over U where $C_i \cap C_j = \phi$ for $i \neq j$. Denote $f_C = \sqcup_{i=1}^{n} f_{iC_i}$, $\varphi_K = \sqcap_{i=1}^{n} f_{iC_i}$.
Where $C = \cup_{i=1}^{n} C_i$ and $K = C_1 \times C_2 \times ... \times C_n$.

Definition 6.2 Let $f_A, g_B \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I$ $\forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Then the soft I-positive region of f_A relative to g_B is defined as follows
$pos^I_{f_A}(g_B) = \bigcup_{b \in B} \underline{apr}_I g(b) = \bigcup_{b \in B} \{x \in U : \exists e \in A,\ s.t\ x \in f(e) \cap g(d)' \in I\}$.

Definition 6.3 Let $f_{i_{C_i}} \in S(U, E)$ (i=1,2,.....n) such that $f_{i_{C_i}}$ is bijective where $C_i \cap C_j = \phi$ for $i \neq j$. Let g_D be a partition soft set over U where $C \cap D = \phi$. Let $P = (U, \varphi_K)$ be a soft approximation space. Then the triple (U, f_C, g_D) is called a soft decision system, f_C is called the condition bijective soft set and g_D is called the decision partition soft set.
In soft decision system $(U, f_{_C}, g_{_D})$, we have
$pos_{(\varphi,K)}(g, D) = \bigcup_{d \in D} \underline{apr}_P g(d) = \bigcup_{d \in D} \{u \in U : \exists e \in K,\ s.t\ x \in \varphi(e) \subseteq g(D)\}$

Definition 6.4 Let (U, f_C, g_D) be a soft decision system. Let I be an ideal on U such that $\varphi(e) \notin I$ $\forall e \in K$. Let (U, φ_K, I) be a soft approximation space via ideal.
We have
$pos^I_{(\varphi_K)}(g_D) = \bigcup_{d \in D} \underline{apr}_I g(d) = \bigcup_{d \in D} \{x \in U : \exists e \in K,\ s.t\ x \in \varphi(e) \cap g(d)' \in I\}$

Definition 6.5 Let (U, f_C, g_D) be a soft decision system and let $1 \leq j \leq n$. Let I be an ideal on U such that $\varphi(e) \notin I$ $\forall e \in K$. Let (U, φ_K, I) be a soft approximation space via ideal. Then

i) $f_{j_{C_j}}$ is called soft I-dispensable set f_C relative to g_D, if $pos^I_{(\varphi_K)}(g, D) = pos^I_{(\psi_Q)}(g, D)$, where $\psi_{_Q} =$

$\sqcap^{n}_{i=1,i\neq j} f_{iC_i}$. Otherwise f_{jC_j} is called soft I-indispensable set f_C relative to g_D,

ii) f_C is called a soft I-independent set relative to g_D, if every soft bijective set f_{iC_i} of f_C is a soft I-indispensable set relative to g_D. Otherwise, f_C is called a soft I-dependent set relative to g_D

iii) The union of soft I-indispensable sets of f_C relative to g_D is called the I-core of f_C relative to g_D, denoted by $core(f_C, g_D)$.

Definition 6.6 Let (U, f_C, g_D) be a soft decision system and let $1 \leq j \leq n$. Let I be an ideal on U such that $\varphi(e) \notin I\ \forall e \in K$. Let (U, φ_K, I) be a soft approximation space via ideal. Let $k = 1, 2, ..., m$ and $1 \leq J_k \leq n$, denote $f^{*}_{C^*} = \sqcup^{m}_{k=1} f_{j_k C_{j_k}}$, $\varphi^{*}_{K^*} = \sqcap^{m}_{k=1} f_{j_k C_{j_k}}$.
$f^{*}_{C^*}$ is called soft I-relative reduction in (U, f_C, g_D) , if

i) $pos^{I}_{(\varphi_K)}(g_D) = pos^{I}_{(\varphi^{*}_{K^*})}(g_D)$

ii) $f^{*}_{C^*}$ is a soft I-independent set relative to g_D

Definition 6.7 Let $f_A, g_B \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. f_A is said to I-depend on g_B to a degree k $(0 \leq\leq k \leq 1)$, denoted $f_A \Rightarrow_k g_B$, if

$$k = \gamma(f_A, g_B) = \frac{|pos^{I}_{f_A}(g_B)|}{|U|}$$

Accordingly, in a soft decision system (U, f_C, g_D), we have

$$k = \gamma(\varphi_K, g_D) = \frac{|pos^{I}_{\varphi_K}(g_D)|}{|U|}$$

which called the I-dependent degree of condition bijective soft sets in classifying decision partition soft sets. Obviously, we have $0 \leq k \leq 1$.

i) If $k = 1$, then g_D is completely I-dependent on f_C.

ii) If $k = 0$, then g_D is completely I-independent on f_C.

Proposition 6.8 Let $f_A, g_B \in S(U, E)$ and I be an ideal on U such that $f(a) \notin I\ \forall a \in A$. Let (U, f_A, I) be a soft approximation space via ideal. Let $m, m \in N$ and $m \leq n$. Then

$$\gamma(\sqcap^{m}_{i=1} f_{iC_i}, g_D) \leq \gamma(\sqcap^{n}_{i=1} f_{iC_i}, g_D)$$

Proof: Since $\gamma(\varphi_K, g_D) = \frac{|pos^{I}_{\varphi_K}(g_D)|}{|U|} = \frac{|\bigcup_{d\in D} \underline{apr}_I g(d)|}{|U|} = \frac{|\bigcup_{d\in D}\{x\in U: \exists e\in K,\ s.t\ x\in\varphi(e)\cap(g(d))'\in I\}|}{|U|}$,

$\gamma(\varphi^{*}_{K^*}, g_D) = \frac{|pos^{I}_{\varphi^{*}_{K^*}}(g_D)|}{|U|} = \frac{|\bigcup_{d\in D} \underline{apr}_I g(d)|}{|U|} = \frac{|\bigcup_{d\in D}\{x\in U: \exists e\in K^*,\ s.t\ x\in\varphi^*(e)\cap(g(d))'\in I\}|}{|U|}$,
For any $(c_1, c_2, ..., c_n) \in C_1 \times C_2 \times ... \times C_n$, we have
$\varphi(c_1, c_2, ..., c_n) = f_1(c_1) \cap f_2(c_2) \cap, ... \cap f_m(c_m) \cap ... \cap f_n(c_n)$.
Moreover, for any $(c_1, c_2, ..., c_m) \in C_1 \times C_2 \times ... \times C_m$, we have
$\varphi(c_1, c_2, ..., c_m) = f_1(c_1) \cap f_2(c_2) \cap, ... \cap f_m(c_m)$. For $m, n \in N$ and $m \leq n$, $\underline{apr}^{*}_I g(d) \subseteq \underline{apr}_I g(d)$. Thus $\bigcup_{d\in D} \underline{apr}^{*}_I g(d) \subseteq \bigcup_{d\in D} \underline{apr}_I g(d)$. Hence

$$\gamma(\sqcap^{m}_{i=1} f_{iC_i}, g_D) \leq \gamma(\sqcap^{n}_{i=1} f_{iC_i}, g_D)$$

Definition 6.9 Let (U, f_C, g_D) be a soft decision system and let $1 \leq j \leq n$. Let I be an ideal on U such that $\varphi(e) \notin I\ \forall e \in K$. Let (U, φ_K, I) be a soft approximation space via ideal. The I- conditional significance of f_{iC_i} in f_C relative to g_D is denoted and defined as follows

$$s(f_{jC_j}, f_C, g_D) = \gamma(\sqcap^{m}_{i=1} f_{iC_i}, g_D) - \gamma(\sqcap^{m}_{i=1,i\neq j} f_{iC_i}, g_D)$$

This significance shows the decrease of the I-dependent degree of decision partition soft sets when deleting on bijective soft set $f_{j_{C_j}}$ from f_C.

Proposition 6.10 Let (U, f_C, g_D) be a soft decision system and let $1 \leq j \leq n$. Let I be an ideal on U such that $\varphi(e) \notin I \ \forall e \in K$. Let (U, φ_K, I) be a soft approximation space via ideal. Let $1 \leq j \leq n$, then

i) $0 \leq s(f_{j_{C_j}}, f_C, g_D) \leq 1$,

ii) $f_{j_{C_j}}$ is a soft I-indispensable set of f_C iff $s(f_{j_{C_j}}, f_C, g_D) \geq 0$,

iii) $core(f_C, g_D) = \sqcup\{f_{j_{C_j}} : s(f_{j_{C_j}}, f_C, g_D) \geq 0, j = 1, 2, ..., n\}$

Theorem 6.11 Let (U, f_C, g_D) be a soft decision system and let $1 \leq j \leq n$. Let I be an ideal on U such that $\varphi(e) \notin I \ \forall e \in K$. Let (U, φ_K, I) be a soft approximation space via ideal. Let $k = 1, 2, ..., m$ and $1 \leq j_k \leq n$, denote
$f^*_{C^*} = \sqcup_{k=1}^{m} f_{j_k C_{j_k}}$, $\varphi^*_{\underset{K}{*}} = \sqcap_{k=1}^{m} f_{j_k C_{j_k}}$
Where $C^* = \cup_{j_k=1}^{m} C_i$ and $K^* = C_{j_1} \times C_{j_2} \times ... \times C_{j_m}$.
If $\gamma(\varphi^*_{\underset{K}{*}}, g_D) = \gamma(\varphi_K, g_D)$ and $s(f_{j_{C_j}}, f_C, g_D) \geq 0$, then $f^{**}_{\ C}$ is I-relative reduction of (U, f_C, g_D).

Definition 6.12 Let (U, f_C, g_D) be a soft decision system and let $1 \leq j \leq n$. Let I be an ideal on U such that $\varphi(e) \notin I \ \forall e \in K$. Let (U, φ_K, I) be a soft approximation space via ideal. Let $e \in K$, $d \in D$. The soft I-rough membership function of $\varphi(e)$ relative to $g(d)$ is denoted and defined by

$$\xi(\varphi(e), g(d)) = \frac{|\varphi(e) \cap g(d)|}{|\varphi(e)|}$$

Theorem 6.13 Let (U, f_C, g_D) be a soft decision system and let $1 \leq j \leq n$. Let I be an ideal on U such that $\varphi(e) \notin I \ \forall e \in K$. Let (U, φ_K, I) be a soft approximation space via ideal. Let $k = 1, 2, ..., m$ and $1 \leq j_k \leq n$, denote
$f^*_{C^*} = \sqcup_{k=1}^{m} f_{j_k C_{j_k}}$, $\varphi^*_{\underset{K}{*}} = \sqcap_{k=1}^{m} f_{j_k C_{j_k}}$
Where $C^* = \cup_{j_k=1}^{m} C_i$ and $K^* = C_{j_1} \times C_{j_2} \times ... \times C_{j_m}$. Let $f^*_{C^*}$ is I-relative reduction of (U, f_C, g_D), then the I-multi-attribute decision rule induced by $f^*_{C^*}$ in (U, f_C, g_D) is

$$if\ e,\ then\ d(\xi(\varphi^*(e), g(d)))$$

where $e \in K^*$, $d \in D$ and $\xi(\varphi^*(e), g(d))$ denotes the soft I-rough membership function of $\varphi^*(e)$ relative to $g(d)$, which expresses the support degree of rules.

Example 6.14 (Acute Coronary Syndrome) Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ is a universe which is a set of six patients suffered from chest pain that raise the possibility of Acute coronary syndrome (Acs). We want to decide which type of ACS according to symptoms, ECG and cardiac enzymes because the way of management differ completely between the 3 types. Let $C = \cup_{i=1}^{3} C_i$ denotes the attributes set where C_1 stands for symptoms, C_2 stands for ECG and C_3 stands for cardiac enzymes. The value sets of these attributes are

$C_1 = \{chest\ pain\ at\ rest(increasing\ exertion),\ Pallor\ and\ sweeting, vomiting\ withchest\ pain\}$

$C_2 = \{ST\ segment\ elevation,\ TS\ segment\ elevation\ with\ or\ without\ T\ wave\ inversion\}$

$C_3 = \{highly\ elevated\ Troponin, CKMB\}$. And $D = \{NSTEMI, STEMI\}$ describes the Acute coronary syndrome STEMI AND NSTEMI.

Suppose that the six patients are characterized by the condition bijective soft set $\cup_{i=1}^{3} C_i$ and the disease (NSTEMI or STEMI) is characterized by the decision partition soft set g_D.
The mapping of each bijective soft set over U is defined as follows:
$f_1(increasing\ exertion) = \{x_1, x_6\}$, $f_1(Pallor\ and\ sweeting) = \{x_2, x_3, x_5\}$, $f_1(vomiting\ withchest\ pain) =$

Table 4: Tabular representation of φ_K

	x_1	x_2	x_3	x_4	x_5	x_6
e_1	1	0	0	0	0	0
e_2	0	1	1	0	0	0
e_3	0	0	0	1	0	0
e_4	0	0	0	0	1	0
e_5	0	0	0	0	0	1

$\{x_4\}$,
$f_2(ST\ segment\ elevation) = \{x_1, x_2, x_3\}$, $f_2(TS\ segment\ elevation\ with\ or\ without\ T\ wave\ inversion) = \{x_4, x_5, x_6\}$,
$f_3(highly\ elevated\ Troponin) = \{x_1, x_2, x_3, x_4\}$, $f_3(Mildly\ elevated\ Troponin) = \{x_5, x_6\}$.
The mapping of the decision partition soft set over U is defined as follows:
$g(NSTEMI) = \{x_1, x_3, x_6\}$, $g(STEMI) = \{x_2, x_4, x_5\}$.
Then we can view each bijective soft set f_{iC_i} as a collection of approximations as follows:
$f_{1C_1} = \{increasing\ exertion = \{x_1, x_6\}, Pallor\ and\ sweeting = \{x_2, x_3, x_5\}, vomiting\ with\ chest\ pain = \{x_4\}\}$
$f_{2C_2} = \{ST\ segment\ elevation = \{x_1, x_2, x_3\}, TS\ segment\ depression\ with\ or\ without\ T\ wave\ inversion = \{x_4, x_5, x_6\}\}$
$f_{3C_3} = \{highly\ elevated\ Troponin = \{x_1, x_2, x_3, x_4\}, Mildly\ elevated\ Troponin = \{x_5, x_6\}\}$
Similarly, $g_D = \{NSTEMI = \{x_1, x_3, x_6\}, STEMI = \{x_2, x_4, x_5\}\}$.
Denote

$$f_C = \sqcup_{i=1}^{3} f_{iC_i}, \qquad \varphi_K = \sqcap_{i=1}^{3} f_{iC_i}$$

where $C = \cup_{i=1}^{3} C_i$ and $K = \cap_{i=1}^{3} C_i$.
Let $e_i \in K$, then
$e_1 = \{increasing\ exertion\ and\ ST\ segment\ elevation\ and\ highly\ elevated\ Troponin\}$
$e_2 = \{Pallor\ and\ sweeting\ and\ ST\ segment\ elevation\ and\ highly\ elevated\ Troponin\}$
$e_3 = \{vomiting\ with\ chest\ pain\ and\ TS\ segment\ depression\ with\ or\ without\ T\ wave\ inversion\ \ highly\ elevated\ Troponin\}$
$e_4 = \{Pallor\ and\ sweeting\ and\ TS\ segment\ depression\ with\ or\ without\ T\ wave\ inversion\ \ Mildly\ elevated\ Troponin\}$
$e_5 = \{increasing\ exertion\ and\ TS\ segment\ depression\ with\ or\ without\ T\ wave\ inversion\ and\ Mildly\ elevated\ Troponin\}$
$\varphi(e_1) = \{x_1\}$, $\varphi(e_2) = \{x_2, x_3\}$, $\varphi(e_3) = \{x_4\}$,
$\varphi(e_4) = \{x_5\}$, $\varphi(e_5) = \{x_6\}$.
The tabular representation of φ_K is given in Table 4

Thus (U, f_C, g_D) is a soft decision system on how to dignose NSTEMI OR STEMI diseases. Let I be an ideal on U defined as follows
$I = \{\phi, \{x_2\}, \{x_5\}, \{x_2, x_5\}\}$, so (U, φ_K, I) is a soft approximation space via ideal. Hence
$\underline{apr}_I g(highly\ elevated\ Troponin) = \{x_1, x_3, x_6\}$ and $\underline{apr}_I g(mildly\ elevated\ Troponin) = \{x_4, x_5\}$.
Therefore $pos^I_{(\varphi_K)}(g_D) = \{x_1, x_3, x_4, x_5, x_6\}$
Denote $\varphi_{1k_1} = f_{1C_1} \sqcap f_{2C_2}$, $\varphi_{2k_2} = f_{1C_1} \sqcap f_{3C_3}$, $\varphi_{3k_3} = f_{2C_2} \sqcap f_{3C_3}$. We have
$pos^I_{(\varphi_{1K_1})}(g_D) = pos^I_{(\varphi_{2K_2})}(g_D) = pos^I_{(\varphi_K)}(g_D) = \{x_1, x_3, x_4, x_5, x_6\}$, $pos^I_{(\varphi_{3K_3})}(g_D) = \{x_1, x_3, x_4, x_6\}$.
But $pos^I_{(f_{1C_1})}(g_D) = \{x_1, x_3, x_4, x_6\} \neq pos^I_{(\varphi_K)}(g_D)$,
$pos^I_{(f_{3C_3})}(g_D) = \{x_6\} \neq pos^I_{(\varphi_K)}(g_D)$.
Therefore $f_{1C_1} \sqcup f_{2C_2}$ and $f_{1C_1} \sqcup f_{3C_3}$ are both I-relative reductions in (U, f_C, g_D).
The I-dependent degree of the decision partition soft set g_D upon the condition bijective soft set f_C

$$k = \gamma(\varphi_K, g_D) = \frac{|pos^I_{\varphi_K}(g_D)|}{|U|} = \frac{|\{x_1, x_3, x_4, x_5, x_6\}|}{|U|} = \frac{5}{6}$$

In the following we will give an algorithm for I-multi-attribute decision rule.
Algorithm:

Step 1. Construct a soft decision system (U, f_C, g_D)

Step 2. Calculate the I-dependent degree of g_D upon $\sqcap_{i=1,i\neq j}^{n} f_{iC_i}$ (j=1,2,...n)

Step 3. Calculate each I-conditional significance of $f_{j_{C_j}}$ relative to g_D by

Step 4. Find I-$core(f_C, g_D)$ by

Step 5. Find I-relative reductions in (U, f_C, g_D) by

(1) If $\gamma(core(f_C, g_D), g_D) = \gamma(f_C, g_D)$, then $core(f_C, g_D)$ is I-relative reduction in (U, f_C, g_D). In this case, the process stops. Otherwise, it continuous 2

(2) Denote $core(f_C, g_D) = \sqcap_{k=1}^{m} f_{j_k C_{j_k}}$, where $k = 1, 2, ..., m$ and $1 \leq j_k \leq n$

(a) Calculate the I-conditional significance of each bijective soft set f_{iC_i} $i \neq j_k$ about $\sqcup_{k=1}^{m} f_{j_k C_{j_k}}$ relative to g_D by

(b) Select f_{iC_i} with maximal I-conditional significance one by one. If there are many soft sets with the same maximal significance, we choose the attribute set containing the most elements. So $core(f_C, g_D) \sqcup f_{iC_i}$ is I-relative reduction in (U, f_C, g_D)

Step 6. Obtain I-decision rules by I-relative reduction in soft decision system (U, f_C, g_D).

7. Conclusion

In this paper, we have proposed the new concept of soft rough sets via ideal. We presented important properties of soft rough approximations via ideal based on soft approximation spaces via ideal, giving interesting examples. The accuracy measure is one of the ways of characterizing soft rough theory. Our approach makes the accuracy measures higher than the existing approximations. Soft rough relations via ideal were discussed. We researched relationships among soft sets, soft rough sets via ideal and topologies, obtained the structure of soft rough sets via ideal. Furthermore, we examined the relationship between soft rough sets via ideal and rough sets via ideal, and compared these two different models.

References

[1] H. Aktas, N. Cagman, Soft sets and soft groups, Inf. Sci. 77(2007) 2726-2735.

[2] Z. Bonikowski , E. Bryniarski , U. Wybraniec, Extensions and intentions in the rough set theory, Inf. Sci. 107(1998) 149-167.

[3] E. Bryniarski, A calculus of a rough set of the first order, Bull. Pol. Acad. Sci. 16(1989) 71-77.

[4] B. Davvaz, Roughness based on fuzzy ideals, Inf. Sci. 176 (2006) 2417- 2437.

[5] D. Dubois , H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. Gen. Syst. 17(2-3)(1990) 191-209.

[6] F. Feng, C. Li, B. Davvaz and M. Irfan Ali, Soft sets combined with fuzzy sets and rough sets:a tentative approach, Soft Comput. 14 (2010) 899-911.

[7] F. Feng, X. Liu, F. L. Violeta and J. B. Young, Soft sets and soft rough sets, Inf. Sci. 181 (2011) 1125-1137.

[8] F. Feng, Soft rough sets applied to multicriteria group decision making, Ann. Fuzzy Math. Inf. 2 (2011) 69-80.

[9] K. Gong, Z.Xiao, X.Zhang, The bijective soft set with its operations, computers and mathematics with applications, 60(2010),2270-2278.

[10] J. Järvinen, On the structure of rough approximations, Fundamenta Informaticae 50(2002) 135-153.

[11] F. Karaaslan, Soft Classes and Soft Rough Classes with Applications in Decision Making, Mathematical Problems in Engineering, 2016, ID 1584528, 11 pages. Journal of applied mathematics, 2013, ID 241485, 15 pages.

[12] GL. Liu, Generalized rough sets over fuzzy lattices, Inf. Sci. 178 (2008) 1651-1662.

[13] JS. Mi, W.Zhang, An axiomatic characterization of a fuzzy generalization of rough sets, Inf. Sci. 160 (2004) 235-249

[14] P. K. Maji, A. R. Roy and R. Biswas, Fuzzy soft sets, J. Fuzzy Math. 9 (2001) 589-602.

[15] P. K. Maji, R. Biswas and A. R. Roy, Intuitionistic fuzzy soft sets, J. Fuzzy Math. 9 (2001) 677-692.

[16] P. K. Maji, A. R. Roy and R. Biswas, Soft set theory, Comput. Math. Appl.45 (2003) 555562.

[17] D. Molodtsov, Soft set theory-First results, Comput. Math. Appl. 37 (1999) 19-31.

[18] D. Molodtsov, The theory of soft sets, URSS Publishers, Moscow (in Russian), 2004.

[19] Heba I. Mustafa, Soft rough approximation operators on a complete atomic Boolean lattice. Mathematical Problems in Engineering, 2013, Article ID 486321, 11 pages.

[20] H. P. Nguyen and V. Kreinovich, Fuzzy logic and its applications in medicine, International Journal of Medical Informatics 62 (2001) 165-173.

[21] Z. Pawlak (1991) Rough sets: theoretical aspects of reasoning about data. Kluwer Academic Publishers, Dordrecht.

[22] Z. Pawlak , A. Skowron, Rudiments of rough sets, Inf. Sci. 177(2007) 3-27.

[23] Z. Pawlak , A. Skowron, Rough sets: some extensions, Inf. Sci. 177(2007) 28-40.

[24] DW. Pei, A generalized model of fuzzy rough sets, Int. J. Gen. Syst. 34(2005) 603-613.

[25] A. Radzikowska, E. Kerre, Fuzzy rough sets based on residuated lattices. In: Peters JF, Skowron A, Dubois D, Grzymala-Busse JW, Inuiguchi M, Polkowski L (eds) Transactions on rough sets II, LNCS3135,(2004) pp 278-296.

[26] M. Shabir, M. I. Ali and T. Shaheen, Another approach to soft rough sets, Knowledge-Based Systems 40 (2013) 72-80.

[27] O. A. Tantawy , H. I. Mustafa, On rough approximations via ideal, Inf. Sci. 251(2013) 114-125.

[28] QM. Xiao , Li QG, Guo LK, Rough sets induced by ideals in lattices, Inf. Sci. 271(2014) 82-92.

[29] W. Xu, J. Ma, S. Y. Wang and G. Hao, Vague soft sets and their properties, Comput. Math. Appl. 59 (2010) 787-794.

[30] X. B. Yang, T. Y. Lin, J. Y. Yang, Y. Li and D. J. Yu, Combination of interval valued fuzzy set and soft set, Comput. Math. Appl. 58 (2009) 521-527.

[31] Y. Yao, Two views of the theory of rough sets in finite universes, International Journal of Approximate Reasoning 15(1996a)291-317.

[32] Y. Yao, T. Lin, Generalization of rough sets using modal logic. Intell. Autom Soft Comput. (1996b) 103-120.

[33] Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, Inf. Sci. 101 (1998a) 239-259.

[34] L. A. Zadeh, Fuzzy sets, Inform. Control. 8 (1965) 338-353.

[35] XH. Zhang , B. Zhou, P. Li, A general frame for intuitionistic fuzzy rough sets, Inf. Sci. 216(2012) 34-49.

[36] L. Zhaowen, Q. Bin, and c. Zhangyong, Soft rough approximation operatoes and related results, Journal of applied mathematics, 2013, ID 241485, 15 pages.

[37] W. Zhu, Generalized rough sets based on relations, Inf. Sci. 177 (2007) 4997-5011.

[38] W. Zhu, Relationship among basic concepts in covering-based rough sets, Inf. Sci. 179(2009) 2478-2486.