ON THE SOLUTIONS OF SOME SYSTEMS OF DIFFERENCE EQUATIONS

H. EL-METWALLY[1], E.M. ELABBASY[1*], AND A. ESHTIBA[2]

ABSTRACT. In this paper, we investihate the dynamical behavior of the positive solutions of the following system of difference equations

$$u_{n+1} = \frac{au_{n-1}}{b + cv_{n-3}^{p}w_{n-1}^{p_1}}, v_{n+1} = \frac{dv_{n-1}}{e + fw_{n-3}^{q}u_{n-1}^{q_1}}, w_{n+1} = \frac{gw_{n-1}}{h + Iu_{n-3}^{r}v_{n-1}^{r_1}},$$

for $n \in \mathbb{N}_0$, where the initial conditions u_{-i}, v_{-i}, w_{-i} $(i = 0, 1, 2, 3)$ are non-negative real numbers and the parameters $a, b, c, d, e, f, g, h, I, p, q, r$ are positive real numbers.

1. INTRODUCTION

The theory of discrete dynamic of systems of difference equations developed greatly during the last thirty years of the twentieth century. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics, psychology. See, for example, [2-8,10,12-16].

Cinar [1] investigated the periodicity of the positive solutions of the system

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}.$$

Kurbanli el al.[9] studied the system of two nonlinear difference equation

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}.$$

In this study, we investigate the dynamic behavior of the positive solutions of the following system of difference equations.

(1.1)

$$u_{n+1} = \frac{au_{n-1}}{b + cv_{n-3}^{p}w_{n-1}^{p_1}}, v_{n+1} = \frac{dv_{n-1}}{e + fw_{n-3}^{q}u_{n-1}^{q_1}}, w_{n+1} = \frac{gw_{n-1}}{h + Iu_{n-3}^{r}v_{n-1}^{r_1}}, n \in \mathbb{N}_0$$

where the initial conditions u_{-i}, v_{-i}, w_i $(i = 0, 1, 2, 3)$ are non-negative real numbers and the parameters $a, b, c, d, e, f, g, h, I, p, q, r$ are positive real numbers.

Consider the difference equation

$$X_{n+1} = H(X_n), \quad n = 0, 1, ... \tag{1.2}$$

where $X_n \in R^n$ and $H \in C^1[R^{k+1}, R^{k+1}]$. Then the linearized equation associated with Eq.(1.2) is given by

$$Y_{n+1} = AY_N, \quad n = 0, 1, ...,$$

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where A is the Jacobian matrix $DH(\overline{X})$ of the function H evaluated at the equilibruim $\overline{X}$.

Theorem A [11]: Let $\overline{X}$ be an equilibruim point of Eq.(1.2) and assume that H is a C^1 function in R^{k+1}. Then the following statements are true:

(a) If all the eigenvalues of the Jacobian matrix $DH(\overline{X})$ lie in the open unit disk$|\lambda| < 1$, then the equilibruim $\overline{X}$ of Eq.(1.2) is asymptotically stable.

(b) If at least one eigenvalues of the Jacobian matrix $DH(\overline{X})$ has absolute value greater than one, then the equilibruim $\overline{X}$ of Eq.(1.2) is unstable.

We will study the following cases:

Case 1. If $p_1 = q_1 = r_1 = 0$.

Case 2. If $p_1 = q_1 = r_1 = 1$.

2. Case 1 System (1.1) when $p_1 = q_1 = r_1 = 0$.

We will investigate the stability of the two equilibrium points of System (1.1) when $p_1 = q_1 = r_1 = 0$. Then from System (1.1) we get

$$u_{n+1} = \frac{au_{n-1}}{b + cv_{n-3}^p}, v_{n+1} = \frac{dv_{n-1}}{e + fw_{n-3}^q}, w_{n+1} = \frac{gw_{n-1}}{h + Iu_{n-3}^r}, \quad n \in \mathbb{N}_0. \tag{2.1}$$

By the change of variables $u_n = (\frac{h}{I})^{\frac{1}{r}} x_n, v_n = (\frac{b}{c})^{\frac{1}{p}} y_n, w_n = (\frac{e}{f})^{\frac{1}{q}} z_n$. System (2.1) can be rewritten as

$$x_{n+1} = \frac{\alpha x_{n-1}}{1 + y_{n-3}^p}, \quad y_{n+1} = \frac{\beta y_{n-1}}{1 + z_{n-3}^r}, \quad z_{n+1} = \frac{\gamma y_{n-1}}{1 + x_{n-3}^q}, \quad n \in \mathbb{N}_0 \tag{2.2}$$

where $\alpha = \frac{a}{b}, \beta = \frac{g}{h}, \gamma = \frac{d}{e}$.

In this section, we investigate the stability of the two equilibruim points of System (2.2). When $\alpha, \beta, \gamma \in (0, 1)$, it is easy to see that $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ is the unique equilibrium point of System (2.2). When $\alpha, \beta, \gamma \in (1, \infty)$, the unique positive equilibrium point of System (2.2) is given by $(\overline{x}_2, \overline{y}_2, \overline{z}_2) = ((\gamma - 1)^{\frac{1}{q}}, (\alpha - 1)^{\frac{1}{p}}, (\beta - 1)^{\frac{1}{r}})$.

Theorem 1. *The following statements hold:*

(i) If $\alpha, \beta, \gamma \in (0, 1)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of System (2.2) is locally asymptotically stable.

(ii) If $\alpha \in (1, \infty)$ *or* $\beta \in (1, \infty)$ *or* $\gamma \in (1, \infty)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of System (2.2) is unstable.

(iii) If $\alpha, \beta, \gamma \in (1, \infty)$, then the positive equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2) = ((\gamma - 1)^{\frac{1}{q}}, (\alpha - 1)^{\frac{1}{p}}, (\beta - 1)^{\frac{1}{r}})$ of System (2.2) is unstable.

Proof. We will rewrite System (2.2) in the form

$$X_{n+1} = F(X_N), \tag{2.3}$$

where $X_n = (x_n, ..., x_{n-3}, y_n, ..., y_{n-3}, z_n, ..., z_{n-3})^T$ and the map F is given by

$$F\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ s_0 \\ s_1 \\ s_2 \\ s_3 \\ k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} \frac{\alpha t_1}{1+s_3^p} \\ t_0 \\ t_1 \\ t_2 \\ \frac{\beta s_1}{1+k_3^r} \\ s_0 \\ s_1 \\ s_2 \\ \frac{\gamma k_1}{1+t_3^q} \\ k_0 \\ k_1 \\ k_2 \end{pmatrix}.$$

The linearized system of (2.3) about the equilibrium point $\overline{X} = (0, ..., 0)^T$ is given by

$$X_{n+1} = J_F(\overline{X}_0)X_n,$$

where

$$J_F(\overline{X}_0) = \begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus the characteristic equation of $J_F(\overline{X}_0)$ is given by

$$\lambda^6(\lambda^2 - \alpha)(\lambda^2 - \beta)(\lambda^2 - \gamma) = 0. \tag{2.4}$$

Then we have the following:

(i) If $\alpha, \beta, \gamma \in (0, 1)$, all the roots of the Eq.(2.4) lie inside the open unit disk $|\lambda| < 1$. So, the unique equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of System (2.2) is locally asymptotically stable.

(ii) It is clearly that if $\alpha \in (1, \infty)$ *or* $\beta \in (1, \infty)$ *or* $\gamma \in (1, \infty)$, then some roots of Eq.(2.4) have absolute value greater that one. Thus, the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of System (2.2) is unstable.

(iii) The linearized system of (2.3) about the positive equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2)$ is given by $X_{n+1} = J_F(\overline{X}_{\alpha,\beta,\gamma})X_n$, where

$$x_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \\ z_n \\ z_{n-1} \\ z_{n-2} \\ z_{n-3} \end{pmatrix}, \quad J_F(\overline{x}_0) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where

$$A = -\frac{p(\alpha -)^{\frac{p-1}{p}}(\beta - 1)^{\frac{1}{r}}}{\alpha}, \; B = -\frac{r(\gamma -)^{\frac{1}{q}}(\beta - 1)^{\frac{r-1}{r}}}{\beta}, \; and \;\; C = -\frac{q(\alpha -)^{\frac{1}{p}}(\gamma - 1)^{\frac{q-1}{q}}}{\gamma}.$$

The characteristic equation of $J_F(\overline{X}_{\alpha,\beta,\gamma})$ is given by

$$p(\lambda) = \lambda^{12} - 3\lambda^{10} + 3\lambda^8 - \lambda^6 - rpq\frac{(\alpha - 1)(\beta - 1)(\gamma - 1)}{\alpha\beta\gamma}.$$

Now

$$p(1) = -rpq\frac{(\alpha - 1)(\beta - 1)(\gamma - 1)}{\alpha\beta\gamma} < 0 \quad \text{and} \quad \lim_{\lambda\to\infty} p(\lambda) = \infty.$$

Then $p(\lambda)$ has at least one root in the interval $(1, \infty)$. So, by Theorem A if $\alpha, \beta, \gamma \in (1, \infty)$, then the positive equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2) = ((\gamma - 1)^{\frac{1}{q}}, (\alpha - 1)^{\frac{1}{p}}, (\beta - 1)^{\frac{1}{r}})$ of System (2.2) is unstable. This completes the proof. □

Theorem 2. *If $\alpha, \beta, \gamma \in (0, 1)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of System (2.2) is globally asymptotically stable.*

Proof. We proved in Theorem 1 that if $\alpha, \beta, \gamma \in (0, 1)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of System (2.2) is locally asymptotically stable. Hence, it suffices to show that

$$\lim_{n\to\infty} (x_n, y_n, z_n) = (0, 0, 0).$$

We have from System (2.2) that

$$\begin{aligned} 0 &\le x_{n+1} = \frac{\alpha x_{n-1}}{1 + y_{n-3}^p} \le \alpha x_{n-1}, \; 0 \le y_{n+1} = \frac{\beta y_{n-1}}{1 + z_{n-3}^r} \le \beta y_{n-1}, \\ 0 &\le z_{n+1} = \frac{\gamma z_{n-1}}{1 + x_{n-3}^q} \le \gamma z_{n-1}, \quad for \; n \in \mathbb{N}_0. \end{aligned}$$

Then it follows by induction that

$$0 \le x_{2n-i} \le \alpha^n x_{-i}, \; 0 \le y_{2n-i} \le \beta^n y_{-i}, \; 0 \le z_{2n-i} \le \gamma^n z_{-i}, \tag{2.5}$$

where x_{-i}, y_{-i}, z_{-i} $(i = 0, 1)$ are the initial conditions. Consequently, by taking limits of inequalities in (2.5) when $\alpha, \beta, \gamma \in (0, 1)$, we get $\lim_{n\to\infty} (x_n, y_n, z_n) = (0, 0, 0)$. This completes the proof. □

Example 1. *Figure (1) shows the global attractivity of the zero equilibrium point* $\overline{x}$ *of System (2.2) for the values* $\alpha = .9, \beta = .2, \gamma = .5$ *and* $p = 2,\ q = .3,\ r = 5$ *whenever* $x_{-3} = 1.04,\ x_{-2} = 2.6,\ x_{-1} = 1.02,\ x_0 = 3.04,\ y_{-3} = 1.3,\ y_{-2} = 3.9,\ y_{-1} = .4,\ y_0 = 1.2,\ z_{-3} = 1.5,\ z_{-2} = 2.3,\ z_{-1} = .9,$ *and* $z_0 = 0.006$.

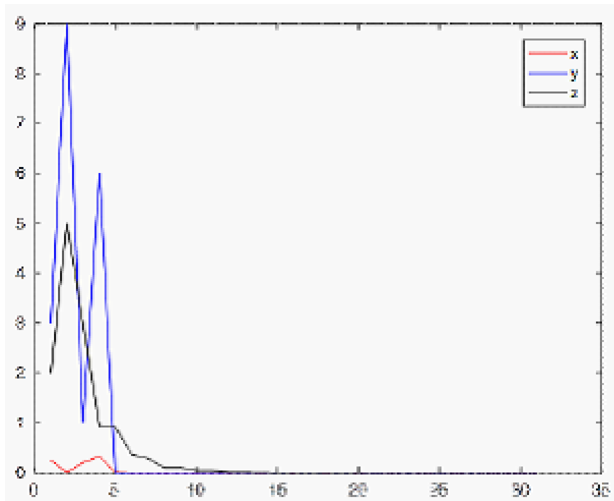

Figure (1)

Theorem 3. *If* $\alpha = \beta = \gamma = 1$, *then every solution of System (2.2) tends a period two solution.*

Proof. We get from System (2.2)

$$x_{2n+1} - x_{2n-1} = -\frac{x_{2n-1}z_{n-3}^{p}}{1+z_{n-3}^{p}} \leq 0,\ y_{2n+1} - y_{2n-1} = -\frac{y_{2n-1}x_{n-3}^{q}}{1+x_{n-3}^{q}} \leq 0,$$

$$z_{2n+1} - z_{2n-1} = -\frac{z_{2n-1}y_{n-3}^{r}}{1+y_{n-3}^{r}} \leq 0.$$

and

$$x_{2n+2} - x_{2n} = -\frac{x_{2n}z_{2n-2}^{p}}{1+z_{2n-2}^{p}} \leq 0,\ y_{2n+2} - y_{2n} = -\frac{y_{2n}x_{2n-2}^{q}}{1+x_{2n-2}^{q}} \leq 0,$$

$$z_{2n+2} - z_{2n} = -\frac{z_{2n}y_{2n-2}^{r}}{1+y_{2n-2}^{r}} \leq 0.$$

Thus, we get

$$x_{2n+1} \leq x_{2n-1},\ y_{2n+1} \leq y_{2n-1},\ z_{2n+1} \leq z_{2n-1},\ x_{2n+2} \leq x_{2n},\ y_{2n+2} \leq y_{2n},$$

and

$$z_{2n+2} \leq z_{2n}.$$

The sequences $\{(x_{2n-1}, y_{2n-1}, z_{2n-1})\}_{n=-3}^{\infty}$ and $\{(x_{2n}, y_{2n}, z_{2n})\}_{n=-3}^{\infty}$ are non-increasing. Hence, while the odd-index terms tend to one periodic point, the even-index terms tend to another periodic point. This completes the proof. □

Theorem 4. *Assume that* $\alpha = \beta = \gamma = 1$, *then every solution* $\{(x_n, y_n, z_n)\}_{n=-3}^{\infty}$ *of System (2.2) converges to a period two solution. Moreover the sequence* $\{x_n\}$

converges to a period solution of the form

$$..., \varphi, \psi, \varphi, \psi, ...,$$

also the sequence $\{y_n\}$ *converges to a period two solution*

$$..., \gamma, \delta, \gamma, \delta, ...,$$

and the sequence $\{z_n\}$ *converges to a period two solution*

$$..., \lambda, \mu, \lambda, \mu, ...,$$

and the solution has the form

$$\{(0,0,0), (\psi, \delta, \mu), (0,0,0), ...\}.$$

Proof. We have from System (2.2)

$$\begin{aligned} x_{n+1} - x_{n-1} &= -\frac{x_{n-1} y_{n-3}^p}{1 + y_{n-3}^p} \leq 0, \; y_{n+1} - y_{n-1} = -\frac{y_{n-1} z_{n-3}^q}{1 + z_{n-3}^q} \leq 0, \\ z_{n+1} - z_{n-1} &= -\frac{z_{n-1} x_{n-3}^r}{1 + x_{n-3}^r} \leq 0, \end{aligned}$$

which imply that $\{x_n\}$ converges to a period two solution

$$..., \varphi, \psi, \varphi, \psi, ...,$$

also $\{y_n\}$ converges to a period two solution

$$..., \gamma, \delta, \gamma, \delta, ...,$$

and $\{z_n\}$ converges to a period two solution

$$..., \lambda, \mu, \lambda, \mu, ... \; .$$

If we assume that

$$\lim_{n\to\infty} x_{2n} = \varphi, \lim_{n\to\infty} x_{2n+1} = \psi, \lim_{n\to\infty} y_{2n} = \gamma, \lim_{n\to\infty} y_{2n+1} = \delta, \lim_{n\to\infty} z_{2n} = \lambda,$$

and

$$\lim_{n\to\infty} z_{2n+1} = \mu,$$

then we have

$$\varphi = \frac{\varphi}{1+\gamma^p}, \; \psi = \frac{\psi}{1+\gamma^p}, \; \gamma = \frac{\gamma}{1+\lambda^r}, \; \delta = \frac{\delta}{1+\lambda^r}, \; \lambda = \frac{\lambda}{1+\varphi^q}, \; \mu = \frac{\mu}{1+\varphi^q}$$

which implies that $\gamma = \lambda = \varphi = 0$. Then the proof is completed. □

Example 2. *Figure (2) shows that the solutions of System (2.2) tend to a period two solution of System (2.2) for the values* $\alpha = \beta = \gamma = 1$ *and* $p = 3,\ q = 3,\ r = 3$ *whenever* $x_{-3} = 4,\ x_{-2} = 6,\ x_{-1} = 2,\ x_0 = 4,\ y_{-3} = .3,\ y_{-2} = .9,\ y_{-1} = 4,\ y_0 = 2,$

$z_{-3} = .5,\ z_{-2} = 2.3,\ z_{-1} = .9,$ *and* $z_0 = 6.$

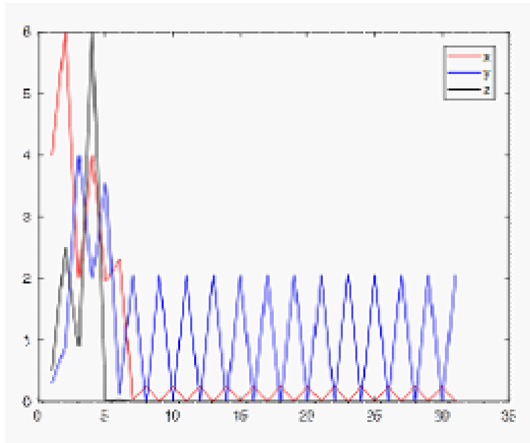

Figure (2)

Here we dell with the oscillation of the positive solutions of System (2.2) about the equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2) = ((\gamma-1)^{\frac{1}{q}}, (\alpha-1)^{\frac{1}{p}}, (\beta-1)^{\frac{1}{r}})$.

Theorem 5. *Let* $\alpha, \beta, \gamma \in (0, \infty)$ *and* $\{(x_{2n}, y_{2n}, z_{2n})\}_{n=-3}^{\infty}$ *be a positive solution of System (2.2). Then,* $\{(x_{2n}, y_{2n}, z_{2n})\}_{n=-3}^{\infty}$ *oscillates about the equilibrium point* $(\overline{x}_2, \overline{y}_2, \overline{z}_2)$*. Moreover, with the possible exception of the first semicycle, every semicycle has length one.*

Proof. Assume that

(i) $x_{-1}, x_{-3} \geq \overline{x}_2,\ x_0, x_{-2} < \overline{x}_2$ *or* $x_{-1}, x_{-2} < \overline{x}_2,\ x_{-3}, x_0 \geq \overline{x}_2,\ y_{-1}, y_{-3} \geq \overline{y}_2,\ y_0, y_{-2} < \overline{y}_2,\ z_0, z_{-2} \geq \overline{z}_2,\ z_{-1}, z_{-3} < \overline{z}_2$

holds. Then we get

$$x_1 = \frac{\alpha x_{-1}}{1+y_{-3}^p} < \overline{x}_2, x_2 = \frac{\alpha x_0}{1+y_{-2}^p} \geq \overline{x}_2, x_3 = \frac{\alpha x_1}{1+y_{-1}^p} < \overline{x}_2, x_4 = \frac{\alpha x_2}{1+y_0^p} \geq \overline{x}_2$$

$$y_1 = \frac{\beta y_{-1}}{1+z_{-3}^r} \geq \overline{y}_2, y_2 = \frac{\beta y_0}{1+z_{-2}^r} < \overline{y}_2, y_3 = \frac{\beta y_1}{1+z_{-1}^r} \geq \overline{y}_2, y_4 = \frac{\beta y_2}{1+z_0^r} < \overline{y}_2$$

$$z_1 = \frac{\gamma z_{-1}}{1+x_{-3}^q} < \overline{z}_2, z_2 = \frac{\gamma z_0}{1+x_{-2}^q} \geq \overline{z}_2, z_3 = \frac{\gamma z_1}{1+x_{-1}^q} < \overline{z}_2, z_4 = \frac{\gamma z_2}{1+x_0^q} \geq \overline{z}_2$$

Then, the result follows by induction. (ii) $x_{-1}, x_{-3} < \overline{x}_2,\ x_0, x_{-2} \geq \overline{x}_2$ *or* $x_{-1}, x_{-2} \geq \overline{x}_2,\ x_{-3}, x_0 < \overline{x}_2,\ y_{-1}, y_{-3} < \overline{y}_2,\ y_0, y_{-2} \geq \overline{y}_2,\ z_0, z_{-2} < \overline{z}_2,\ z_{-1}, z_{-3} \geq \overline{z}_2$. The proof of this case is similarly to case (i) will be omitted. □

In the following theorem, we show the existence of unbounded solutions for System (2.2)

Theorem 6. *If* $\alpha, \beta, \gamma \in (1, \infty)$*, then System (2.2) possesses an unbounded solution.*

Proof. Assume that $\{(x_{2n}, y_{2n}, z_{2n})\}_{n=-3}^{\infty}$ be a solution of System (2.2) with $x_{2n-3} < \overline{x}_2$, $x_{2n-2} \geq \overline{x}_2$, $y_{2n-3} \geq \overline{y}_2$, $y_{2n-2} < \overline{y}_2$, $z_{2n-3} < \overline{z}_2$, *and* $z_{2n-2} \geq \overline{z}_2$ for $n \in \mathbb{N}_0$.Then, we have

$$x_{2n+2} = \frac{\alpha x_{2n}}{1 + y_{2n-2}^p} \geq x_{2n}, y_{2n+1} = \frac{\beta y_{2n-1}}{1 + z_{2n-3}^r} \geq y_{2n-1}, z_{2n+1} = \frac{\gamma z_{2n-1}}{1 + x_{2n-3}^q} \geq z_{2n-1},$$

$$x_{2n+1} = \frac{\alpha x_{2n-1}}{1 + y_{2n-3}^p} < x_{2n-1}, y_{2n+2} = \frac{\beta y_{2n}}{1 + z_{2n-2}^r} < y_{2n}, z_{2n+1} = \frac{\gamma z_{2n}}{1 + x_{2n-2}^q} < z_{2n}.$$

from which it follows that $\lim_{n\to\infty}(x_{2n}, y_{2n-1}, z_{2n-1}) = (\infty, \infty, \infty)$ and $\lim_{n\to\infty}(x_{2n-1}, y_{2n}, z_{2n}) = (0, 0, 0)$.

This completes the proof. □

Example 3. *Figure (3) shows that System (2.2) has unbounded solutions with the values* $\alpha = 1.02, \beta = 1.09, \gamma = 1.05$ *and* $p = q = r = 3$ *whenever* $x_{-3} = 4$, $x_{-2} = 6$, $x_{-1} = 2$, $x_0 = 3$, $y_{-3} = 1.36$, $y_{-2} = 3$, $y_{-1} = 1$, $y_0 = .4$, $z_{-3} = 2$, $z_{-2} = 1.25$, $z_{-1} = 0.23$, *and* $z_0 = 3$.

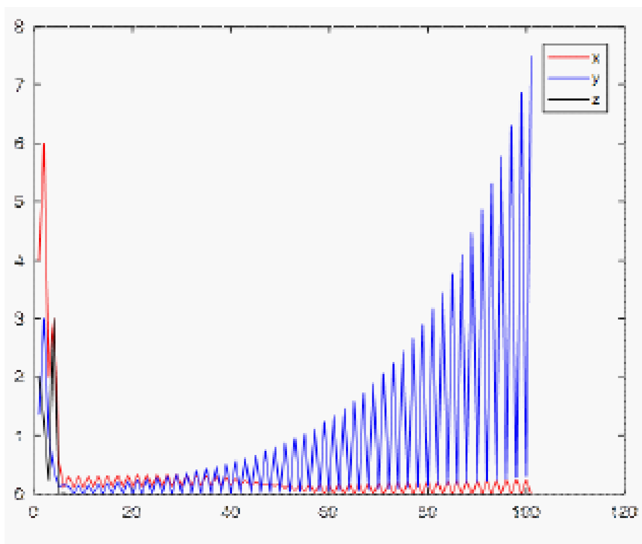

Figure (3)

3. Case 2 System (1.1) when $p_1 = q_1 = 1$.

Now we will investigate the stability of the two equilibrium points of System (1.1) when $p_1 = q_1 = r_1 = 1$. Then from System (1.1) we get

(3.1)

$$u_{n+1} = \frac{a u_{n-1}}{b + c v_{n-3}^p w_{n-1}}, v_{n+1} = \frac{d v_{n-1}}{e + f w_{n-3}^q u_{n-1}}, w_{n+1} = \frac{g w_{n-1}}{h + I u_{n-3}^r v_{n-1}}, \ n \in \mathbb{N}_0$$

By the change of variables $u_n = (\frac{h}{I})^{\frac{1}{r}} x_n, v_n = (\frac{b}{c})^{\frac{1}{p}} y_n, w_n = (\frac{e}{f})^{\frac{1}{q}} z_n$. System (3.1) can be rewritten as

$$(3.2) \quad x_{n+1} = \frac{\alpha x_{n-1}}{1 + s y_{n-3}^{p} z_{n-1}}, \quad y_{n+1} = \frac{\beta y_{n-1}}{1 + t z_{n-3}^{r} x_{n-1}}, \quad z_{n+1} = \frac{\gamma z_{n-1}}{1 + x_{n-3}^{q} y_{n-1}}$$

where $\alpha = \frac{a}{b}, \beta = \frac{d}{e}, \gamma = \frac{g}{h}$, and $s = (\frac{e}{f})^{\frac{1}{q}}, t = (\frac{h}{I})^{\frac{1}{r}}, k = (\frac{b}{c})^{\frac{1}{p}}$.

In this section, we investigate the stability of the two equilibrium points of System (3.2). When $\alpha, \beta, \gamma \in (0,1)$, it is easy to see that $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0,0,0)$ is the unique equilibrium point of System (3.2). When $\alpha, \beta, \gamma \in (1, \infty)$, the unique positive equilibrium point of System (3.2) is $(\overline{x}_2, \overline{y}_2, \overline{z}_2) = ((\frac{\gamma-1}{k})^{\frac{1}{r+1}}, (\frac{\alpha-1}{s})^{\frac{1}{p+1}}, (\frac{\beta-1}{t})^{\frac{1}{q+1}})$.

Theorem 7. *The following statements hold:*

(i) If $\alpha, \beta, \gamma \in (0,1)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0,0,0)$ of System (3.2) is locally asymptotically stable.

(ii) If $\alpha \in (1, \infty)$ *or* $\beta \in (1, \infty)$ *or* $\gamma \in (1, \infty)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0,0,0)$of System (3.2) is unstable.

(iii) If $\alpha, \beta, \gamma \in (1, \infty)$, then the positive equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2)$ of System (3.2) is unstable.

Proof. We rewrite System (3.2) in the form

$$X_{n+1} = F(X_n)$$

where $X_n = (x_n, ..., x_{n-3}, y_n, ..., y_{n-3}, z_n, ..., z_{n-3})^T$ and the map F is given by

$$F \begin{pmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \\ m_0 \\ m_1 \\ m_2 \\ m_3 \\ l_0 \\ l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} \frac{\alpha n_1}{1+s m_3^p l_1} \\ n_0 \\ n_1 \\ n_2 \\ \frac{\beta m_1}{1+t l_3^q n_1} \\ m_0 \\ m_1 \\ m_2 \\ \frac{\gamma l_1}{1+k n_3^r m_1} \\ l_0 \\ l_1 \\ l_2 \end{pmatrix}.$$

The linearized system of (3.3) about the equilibrium point $\overline{X} = (0, ..., 0)^T$ is given by

$$X_{n+1} = J_F(\overline{X}_0) X_n,$$

where

$$J_F(\overline{X}_0)=\begin{pmatrix} 0&\alpha&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\beta&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&\gamma&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix}.$$

Thus the characteristic equation of $J_F(\overline{X}_0)$ is given by

$$\lambda^6(\lambda^2-\alpha)(\lambda^2-\beta)(\lambda^2-\gamma)=0. \tag{3.3}$$

We have the following: (i) If $\alpha,\beta,\gamma\in(0,1)$,all roots of the characteristic equation (3.4) lie inside the open unit disk $\mid\lambda\mid<1$. So, the unique equilibrium point $(\overline{x}_1,\overline{y}_1,\overline{z}_1)=(0,0,0)$ of System (3.2) is locally asymptotically stable.

(ii) If $\alpha\in(1,\infty)$ *or* $\beta\in(1,\infty)$ *or* $\gamma\in(1,\infty)$, then some roots of Eq.(3.4) have absolute values greater than one. Thus, the equilibrium point $(\overline{x}_1,\overline{y}_1,\overline{z}_1)=(0,0,0)$ is unstable.

(iii) The linearized system of (3.3) about the positive equilibrium point $(\overline{x}_2,\overline{y}_2,\overline{z}_2)$ is given by

$$X_{n+1}=J_F(\overline{X}_{\alpha,\beta,\gamma})X_n.$$

where

$$X_n=\begin{pmatrix} x_n\\ x_{n-1}\\ x_{n-2}\\ x_{n-3}\\ y_n\\ y_{n-1}\\ y_{n-2}\\ y_{n-3}\\ z_n\\ z_{n-1}\\ z_{n-2}\\ z_{n-3}\end{pmatrix},\ J_F(\overline{X}_{\alpha,\beta,\gamma})=\begin{pmatrix} 0&A&0&0&0&0&0&B&0&C&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&D&0&0&0&E&0&0&0&0&0&F\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&G&0&H&0&0&0&I&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix},$$

where

$$A=\frac{\alpha t^{\frac{1}{q+1}}}{t^{\frac{1}{q+1}}+(s(\alpha-1)^p)^{\frac{1}{p+1}}(\beta-1)^{\frac{1}{q+1}}},\ B=-\frac{p\alpha s^{\frac{2}{p+1}}t^{\frac{1}{q+1}}(\gamma-1)^{\frac{1}{r+1}}(\beta-1)^{\frac{1}{q+1}}(\alpha-1)^{\frac{p-1}{p+1}}}{k^{\frac{1}{r+1}}(t^{\frac{1}{q+1}}+(s(\alpha-1)^p)^{\frac{1}{p+1}}(\beta-1)^{\frac{1}{q+1}})^2},$$

$$C=-\frac{\alpha t^{\frac{2}{q+1}}(s(\alpha-1)^p)^{\frac{1}{p+1}}(\gamma-1)^{\frac{1}{r+1}}}{k^{\frac{1}{r+1}}(t^{\frac{1}{q+1}}+(s(\alpha-1)^p)^{\frac{1}{p+1}}(\beta-1)^{\frac{1}{q+1}})^2},$$

$$D=-\frac{\beta k^{\frac{2}{r+1}}(t(\beta-1)^q)^{\frac{1}{q+1}}(\alpha-1)^{\frac{1}{p+1}}}{s^{\frac{1}{p+1}}(k^{\frac{1}{r+1}}+(t(\beta-1)^q)^{\frac{1}{q+1}}(\gamma-1)^{\frac{1}{r+1}})^2},\ E=\frac{\beta k^{\frac{1}{r+1}}}{k^{\frac{1}{r+1}}+(t(\beta-1)^q)^{\frac{1}{q+1}}(\gamma-1)^{\frac{1}{r+1}}},$$

$$F=-\frac{\beta q t^{\frac{2}{q+1}} k^{\frac{1}{r+1}\frac{1}{q+1}}(\gamma-1)^{\frac{1}{r+1}}(\beta-1)^{\frac{q-1}{q+1}}(\alpha-1)^{\frac{1}{p+1}}}{s^{\frac{1}{p+1}}(k^{\frac{1}{r+1}}+(t(\beta-1)^{q})^{\frac{1}{q+1}}(\gamma-1)^{\frac{1}{r+1}})^{2}},$$

$$\begin{aligned} G &= -\frac{\gamma r k^{\frac{2}{r+1}} s^{\frac{1}{p+1}}(\gamma-1)^{\frac{r-1}{r+1}}(\beta-1)^{\frac{1}{q+1}}(\alpha-1)^{\frac{1}{p+1}}}{t^{\frac{1}{q+1}}(s^{\frac{1}{p+1}}+(k(\gamma-1)^{r})^{\frac{1}{r+1}}(\alpha-1)^{\frac{1}{p+1}})^{2}}, \\ H &= -\frac{\gamma s^{\frac{2}{p+1}}(k(\gamma-1)^{r})^{\frac{1}{r+1}}(\beta-1)^{\frac{1}{q+1}}}{t^{\frac{1}{q+1}}(s^{\frac{1}{p+1}}+(k(\gamma-1)^{r})^{\frac{1}{r+1}}(\alpha-1)^{\frac{1}{p+1}})^{2}}, \end{aligned}$$

and

$$I=\frac{\gamma s^{\frac{1}{p+1}}}{s^{\frac{1}{p+1}}+(k(\gamma-1)^{r})^{\frac{1}{r+1}}(\alpha-1)^{\frac{1}{p+1}}}.$$

The characteristic equation of $J_F(\overline{X}_{\alpha,\beta,\gamma})$ is given by

$$\begin{aligned} p(\lambda) &= \lambda^{12}-(A+E+I)\lambda^{10}+(EI+AE+AI)\lambda^{8} \\ &\quad -(CG+FH+BD+CHD+AEI)\lambda^{6}+(BDI+AFH+CGE)\lambda^{4}-BFG. \end{aligned}$$

Therefor

$$p(0)=-BFG<0 \quad \text{and} \lim_{\lambda\to\infty}p(\lambda)=\infty.$$

Then $p(\lambda)$ has at least one root in the interval $(1,\infty)$. So by Theorem A we say that if $\alpha,\beta,\gamma\in(0,\infty)$, then the positive equilibrium point $(\overline{x}_2,\overline{y}_2,\overline{z}_2)$ of System (3.2) is unstable. This completes the proof. □

Theorem 8. *If $\alpha,\beta,\gamma\in(0,1)$, then the equilibrium point $(\overline{x}_1,\overline{y}_1,\overline{z}_1)=(0,0,0)$ of System (3.2) is globally asymptotically stable.*

Proof. We proved in Theorem 7 that if $\alpha,\beta,\gamma\in(0,1)$, then the equilibrium point $(\overline{x}_1,\overline{y}_1,\overline{z}_1)=(0,0,0)$ of System (3.2) is locally asymptotically stable. Hence, it suffices to show that

$$\lim_{n\to\infty}(x_n,y_n,z_n)=(0,0,0).$$

We see from System (3.2) that, for $n\in N_0$

$$\begin{aligned} 0 &\le x_{n+1}=\frac{\alpha x_{n-1}}{1+s y_{n-3}^{p} z_{n-1}}\le\alpha x_{n-1},\ 0\le y_{n+1}=\frac{\beta y_{n-1}}{1+t z_{n-3}^{q} x_{n-1}}\le\beta y_{n-1}, \\ 0 &\le z_{n+1}=\frac{\gamma z_{n-1}}{1+k x_{n-3}^{r} y_{n-1}}\le\gamma z_{n-1}. \end{aligned}$$

Then it follows by induction that

$$0\le x_{2n-i}\le\alpha^{n}x_{-i},0\le y_{2n-i}\le\beta^{n}y_{-i},0\le z_{2n-i}\le\gamma^{n}z_{-i}. \tag{3.4}$$

where $x_{-i},y_{-i},z_{-i}(i=0,1)$ are the initial conditions. Consequently, by taking limits of inequalities in (3.5), we get $\lim_{n\to\infty}(x_n,y_n,z_n)=(0,0,0)$. □

Example 4. *Figure (4) shows the global attractivity of the zero equilibrium point $\overline{x}$ of System (3.2) for the values $\alpha=.011$, $\beta=.827$, $\gamma=.021$, $p=.003$, $q=0.01283$, $r=0.343$ and $s=1$, $t=3$, $k=2$ whenever $x_{-3}=1.04$, $x_{-2}=2.6$, $x_{-1}=1.02$,*

$x_0 = 3.04$, $y_{-3} = 1.3$, $y_{-2} = 3.9$, $y_{-1} = .4$, $y_0 = 1.2$, $z_{-3} = 1.5$, $z_{-2} = 2.3$, $z_{-1} = .9$, *and* $z_0 = 0.006$.

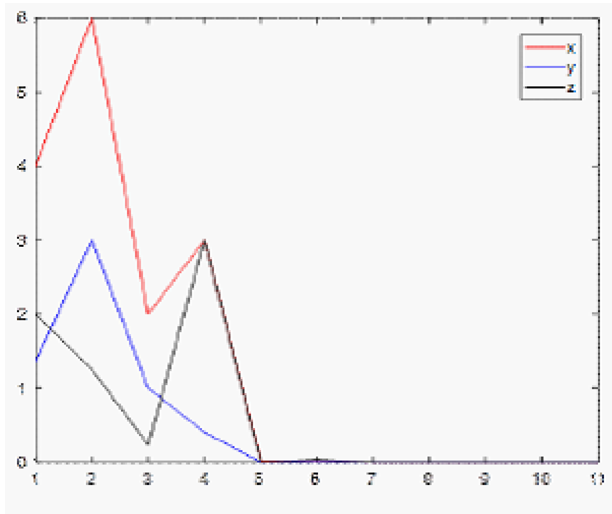

Figure (4)

In the folowing theorem, we investigate the convergence of the period solutions period two of System (3.2).

Theorem 9. *If $\alpha = \beta = \gamma = 1$, then every solution of System (3.2) tends to a period two solution.*

Proof. We get from System (3.2) that

$$x_{2n+1} - x_{2n-1} = -\frac{sx_{2n-1}y_{2n-3}^{p}z_{2n-1}}{1+sy_{2n-3}^{p}z_{2n-1}} \leq 0,\ y_{2n+1} - y_{2n-1} = -\frac{ty_{2n-1}z_{2n-3}^{q}x_{2n-1}}{1+tz_{2n-3}^{q}x_{2n-1}} \leq 0,$$
$$z_{2n+1} - z_{2n-1} = -\frac{ky_{2n-1}x_{2n-3}^{r}z_{2n-1}}{1+kx_{2n-3}^{r}y_{2n-1}} \leq 0$$

and

$$x_{2n+2} - x_{2n} = -\frac{sx_{2n}y_{2n-2}^{p}z_{2n}}{1+sy_{2n-2}^{p}z_{2n}} \leq 0,\ y_{2n+2} - y_{2n} = -\frac{ty_{2n}z_{2n-2}^{q}x_{2n}}{1+tz_{2n-2}^{q}x_{2n}} \leq 0,$$
$$z_{2n+2} - z_{2n} = -\frac{ky_{2n}x_{2n-2}^{r}z_{2n}}{1+kx_{2n-2}^{r}y_{2n}} \leq 0,$$

also

$$x_{2n+2} - x_{2n} = -\frac{sx_{2n}y_{2n-2}^{p}z_{2n}}{1+sy_{2n-2}^{p}z_{2n}} \leq 0,\ y_{2n+2} - y_{2n} = -\frac{ty_{2n}z_{2n-2}^{q}x_{2n}}{1+tz_{2n-2}^{q}x_{2n}} \leq 0,$$
$$z_{2n+2} - z_{2n} = -\frac{ky_{2n}x_{2n-2}^{r}z_{2n}}{1+kx_{2n-2}^{r}y_{2n}} \leq 0.$$

Thus we get

$$x_{2n+1} \leq x_{2n-1},\ y_{2n+1} \leq y_{2n-1},\ z_{2n+1} \leq z_{2n-1},\ x_{2n+2} \leq x_{2n},\ y_{2n+2} \leq y_{2n},$$

and

$$z_{2n+2} \leq z_{2n}.$$

That is , the sequences $\{(x_{2n-1}, y_{2n-1}, z_{2n-1})\}_{n=-3}^{\infty}$ and $\{(x_{2n}, y_{2n}, z_{2n})\}_{n=-3}^{\infty}$ are non-increasing. Hence, while the odd-index terms tend to one periodic point, the even-index terms tend to another periodic point. This completes the proof. □

Example 5. *Figure (5) shows that the solutions of (3.2) tend to a period two solution of System (3.2) for the values* $\alpha = \beta = \gamma = 1$, $p = .3$, $q = .8$, $r = 3$ *and* $s = .09$, $r = 1.54$, $k = .922$ *whenever* $x_{-3} = 4$, $x_{-2} = 6$, $x_{-1} = 2$, $x_0 = 3$, $y_{-3} = 1.36$, $y_{-2} = 3$, $y_{-1} = 1$, $y_0 = .4$, $z_{-3} = 2$, $z_{-2} = 1.25$, $z_{-1} = .23$, *and* $z_0 = 3$.

Example 6. *Figure (6) shows that System (3.2) has an unbounded solution with* $\alpha = 1.02, \beta = 1.09, \gamma = 1.05$, $p = 3$, $q = 3$, $r = 3$ *and* $s = .09$, $r = 1.54$, $k = .922$ *whenever* $x_{-3} = 4$, $x_{-2} = 6$, $x_{-1} = 2$, $x_0 = 3$, $y_{-3} = 1.36$, $y_{-2} = 3$, $y_{-1} = 1$, $y_0 = .4$, $z_{-3} = 2$, $z_{-2} = 1.25$, $z_{-1} = .23$, *and* $z_0 = 3$.

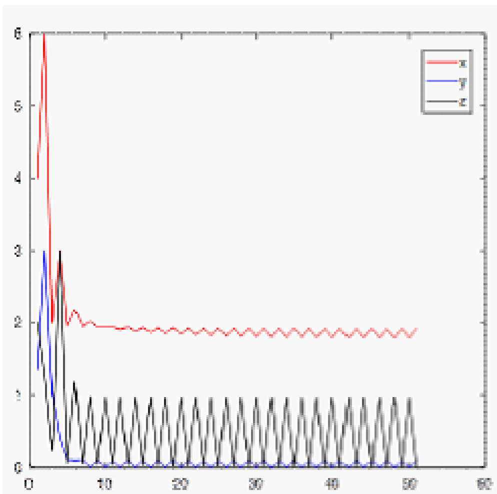

Figure (5)

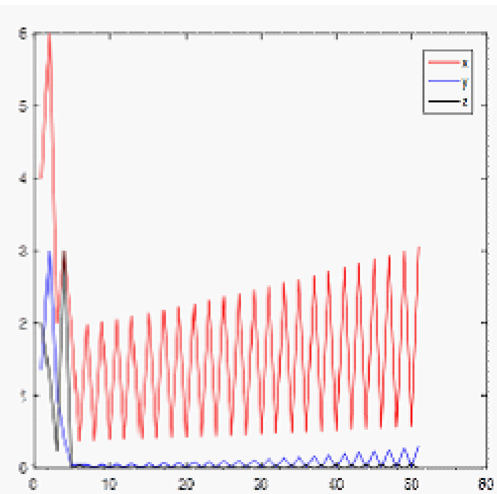

Figure (6)

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[1]Mathematics Department, Faculty of Science, Mansoura University Egypt, [2]Mathematics Department, Faculty of Education, Tripoli University, Libya

E-mail address: [1]eaash69@yahoo.com, [1*]emelabbasy@mans.edu.eg & [2]amnaeshtiba@gmail.com.

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