EXTENSIONS OF VON NEUMANN LOCAL RINGS

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Abstract

This paper introduces an extension of von Neumann local rings (called EVNL-ring). Among other results, we show that $\mathbb{Z}_n$ is an EVNL-ring for all integers $n \geq 2$, the direct product of EVNL-rings is an EVNL-ring and the class of EVNL-rings coincides with the class of C-rings. Also, we study the case when a formal power series ring is an EVNL-ring.

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1 Introduction

In this paper, all rings considered are commutative with identity unless otherwise stated. A ring R is called a pm-ring if every prime ideal is contained in a unique maximal ideal. This class of rings has been studied in many papers, see for example [4], [5], [8] and [9]. A ring R is called TB-ring if the maximal spectrum is a boolean topological space. A ring R is called a von Neumann local ring (VNL-ring) if a or $1-a$ or both has a von Neumann inverse whenever $a \in R$. A ring R is said to be a C-ring if for every $a,\ a' \in R$ with $a + a' = 1$, there exists an idempotent $e \in R$ such that $e \in Ra$ and $e' = 1 - e \in Ra'$.

In [5] Contessa introduced the classes of C-rings, D-rings and VNL-rings as special types of pm-rings . She showed that the class of C-rings coincide with the class of TB-rings. The classes of pm-rings and TB-rings are closed under direct products, however the class of VNL-rings is not closed under direct products. The following diagram shows the relationship between these classes.

$$VNR\text{-}rings \Longrightarrow VNL\text{-}rings \Longrightarrow C\text{-}rings \Longrightarrow pm\text{-}rings.$$

Abu Osba, et al. [1] gave the necessary and sufficient conditions for $\mathbb{Z}_n$, the ring of integers modulo n, to be a VNL-ring. They also showed that the ring of formal power series $R[[x]]$ is a VNL-ring if and only if it is a local ring if and only if the ground ring R is a local ring. They obtained the necessary and sufficient conditions for direct products of a VNL-rings to be a VNL-ring.

Recently, many authors introduced generalizations of VNL-rings, SVNL-rings, n-VNL-rins and GVNL-rings (see [1], [2], [3] and [6]). The main purpose of this paper is to extend the results of Abu Osba et al. in [1] on VNL-rings to EVNL-rings.

We introduce an extension of a VNL-ring, an EVNL-ring, defined as a ring in which $1 - ab$ has a von Neumann inverse for some nonzero $b \in R$ whenever $a \in R$. The trivial examples of EVNL-rings are regular, local and VNL-rings.

In Section 2, we give the basic definitions and properties of EVNL-rings. Among other results, we show that $\mathbb{Z}_n$ is an EVNL-ring for all integers $n \geq 2$. In Section 3, we study the relation between the EVNL-rings and some special classes. The direct product of EVNL-rings is an EVNL-ring. We show that a formal power series ring is an EVNL-ring. Also, we give another characterization of C-rings.

Throughout this paper, $Id(R)$ denotes the set of idempotents of a ring R. $R[[x]]$ is the formal power series ring over R. For $f(x) \in R[[x]]$, we denote by $c(f(x))$ the constant term of $f(x)$. Let $\mathbb{Z}_n$ be the ring of integers modulo n for an integer $n \geq 2$. The annihilator of an element a is denoted by $ann(a)$. $J(R)$ denotes the Jacobson radical of a ring R.

2 Definitions and Examples

In this section, we define EVNL-rings and give examples and basic properties of this ring. An element a of a ring R is called regular if there exist $b \in R$ such that $a = aba$. In this case, we say that a has von Neumann inverse (quasi inverse) b. A ring R is regular if every element in R is regular. In the following we generalize the definition of regular elements.

Definition 1 *An element a of a ring R is said to be extended regular if $1 - ab$ has a von Neumann inverse for some nonzero $b \in R$.*

Clearly all idempotent, regular, nilpotent, unit and zero divisor elements are extended regular elements.

Now, we give the definition of EVNL-rings.

Definition 2 *A ring R is said to be an extended von Neumann local ring (EVNL-ring, for short) if every element in R is extended regular.*

The class of EVNL-rings extends the classes of regular, local and VNL-rings. However, the class of EVNL-rings does not contain the class of integral domains, for example $\mathbb{Z}$ is not an EVNL-ring. Here we give some nontrivial examples of EVNL-rings.

1. Let $R = \mathbb{Z}_{p_1^k p_2 \cdots p_m}$ where p_i is a prime number for all $1 \leq i \leq m$, and $m,\ k$ are positive integers. Then the ring R is a VNL-ring. Moreover R is a local ring if $m = 1$, and R is a VNR-ring if $k = 1$. So 36 is the least positive integer such that $\mathbb{Z}_{36}$ is an EVNL-ring which is not VNL.

2. An integral domain R is an EVNL-ring if and only if for every $a \in R$, $1 - ab$ is a unit for some nonzero element $b \in R$.

3. A ring of matrices over a VNL-ring $\mathbb{Z}_8$, $M_n(\mathbb{Z}_8)$ is an EVNL-ring that is neither a VNR-ring, a local ring nor a VNL-Ring.

4. $\mathbb{Z}_{12} \oplus \mathbb{Z}_{12}$ is an EVNL-ring but not a VNL-ring.

The homomorphic image of a VNL-ring is a VNL-ring, see [1]. However the homomorphic image of an EVNL-ring R is not necessary an EVNL-ring (see Proposition 10 below).

It was shown that $\mathbb{Z}_n$ is a VNL-ring if and only if n is not divided by $(pq)^2$ where p and q are two distinct primes [1, Proposition 2.9]. The aim of the following proposition is to extend this result to EVNL-rings.

Proposition 3 *The ring $\mathbb{Z}_n$ is an EVNL-ring for all n.*

Proof. Since every element of $\mathbb{Z}_n$ is either a unit or a zero divisor, the proof of the proposition follows immediately. ■

Remark 4 *If R is an EVNL-ring which has only the trivial idempotents, then for every nonzero element $a \in R$ we have either a is unit or there exists $b \in R$ such that $1 - ab$ is a unit.*

3 Special classes of EVNL-rings

3.1 Products of EVNL-rings

It is well-known that the direct product of regular rings is a regular ring, while the direct product of local rings is not necessary a local ring. In [5], Contessa showed that the direct product of VNL-rings is not necessary a VNL-ring, for instance, $\mathbb{Z}_8 \oplus \mathbb{Z}_8$ is not a VNL-ring. In [1], the authors answered the question, "When is the direct product of rings a VNL-ring?" They showed that the direct product of rings is a VNL-ring if and only if all those rings are VNR-rings except at most one can be a VNL-ring. So, a natural question arising, what about EVNL-rings? The following theorem answers this question affirmatively.

Theorem 5 *The direct product $R = \prod_{i \in I} R_i$ of a family of rings $\{R_i : i \in I\}$ is an EVNL-ring if and only if R_i is an EVNL-ring for some $i \in I$.*

Proof. Suppose that R is an EVNL-ring, R_i is not an EVNL-ring for each $i \in I$, then each R_i contains an element a_i which is not an extended regular element. Therefore $a = \prod_{i\in I} a_i \in R$ is not extended regular.

On the other hand, assume that R_i is EVNL for some $i \in I$. Hence for every $a_i \in R_i$ there exist $b_i \in R_i$ such that $1 - a_i b_i$ is regluar. Let $a = \prod_{i\in I} a_i \in R$. Now, we define $b = \prod_{j\in I} x_j \in R$ by $x_j = \begin{cases} 0 & \text{if } j \neq i \\ b_j & \text{if } j = i \end{cases}$. Hence $1 - ab$ has a von Neumann inverse, i.e., a is an extended regular element. ■

3.2 Formal Power Series EVNL-Rings

We study the ring of formal power series $R[[x]]$ over EVNL-rings. We will use the following well-known lemmas in the sequel.

Lemma 6 *[5, Proposition 4.1] If an element $a \in R$ is regular, then there exists a unique quasi-inverse, say $a^{(-1)}$, with the following properties:*

1. $aa^{(-1)}a = a$ *and* $a^{(-1)}aa^{(-1)} = a^{(-1)}$.
2. $e = aa^{(-1)}$ *is an idempotent.*
3. $u = 1 - e + a$ *is a unit.*
4. $a = ue$.
5. $Ra = Re$

Lemma 7 *[7] $f(x)$ is invertible in $R[[x]]$ if and only if $c(f(x))$ is invertible in R.*

In [1], the authors proved that $\mathbb{Z}_n[[x]]$ is a VNL-ring if and only if n is a power of some prime. In the following proposition, we generalize this result for EVNL-rings.

Proposition 8 *The formal power series over the ring $\mathbb{Z}_n$, $\mathbb{Z}_n[[x]]$ is an EVNL-ring for all integer $n > 1$.*

Proof. Let $f(x) \in \mathbb{Z}_n[[x]]$. Then the constant term $c(f)$ of $f(x)$ is either a unit or a zero divisor. In case $c(f)$ is unit then $f(x)$ is unit, hence $f(x)$ is an extended regular element. Otherwise there exists a non zero element $b \in \mathbb{Z}_n$ such that $c(1 - bf(x)) = 1$. Therefore $1 - bf(x)$ is regular, i.e. $f(x)$ is an extended regular element. Hence $\mathbb{Z}_n[[x]]$ is an EVNL-ring. ■

Abu Osba, et al. [1] showed that if $R[[x]]$ is a VNL-ring then R and $R[[x]]$ have only two idempotents. Unfortunately, that is not true in case of EVNL rings. For instance, $\mathbb{Z}_{36}[[x]]$ is an EVNL-ring, but $\mathbb{Z}_{36}$ has 9 as an idempotent.

Remark 9 *Let R be a ring. If the set of idempotents of R is $Id(R) = \{0, 1\}$, then every nonzero regular element is a unit. Hence a VNL-ring which has only the trivial idempotents is a local ring.*

In [1], Abu Osba et al. defined a ring R to be an SVNL-ring if whenever $< S >= R$ for some subset $S \subset R$, at least one element of S has a von Neumann inverse. They showed that the ring of formal power series $R[[x]]$ is a VNL-ring (SVNL-ring) if and only if R is a VNL-ring (SVNL-ring) and $Id(R[[x]]) = Id(R) = \{0, 1\}$. Using the above remark, we get that the class of VNL-formal power series rings coincides with the class of local formal power series rings.

Proposition 10 *For any ring R, the formal power series $R[[x]]$ is an EVNL-ring.*

Proof. Let R be any ring, $f(x) = \sum_i a_i x^i$ be an element of $R[[x]]$. Then $1 - xf(x)$ is unit. So $R[[x]]$ is an EVNL-ring. ■

3.3 C-rings

C-rings were first introduced in [5] to associate with any ring a universal pm-ring.

Definition 11 *[5, Definition 3.1] A ring R is said to be a C-ring if for every $a, a' \in R$ with $a + a' = 1$, there exists an idempotent $e \in R$ such that $e \in Ra$ and $e' = 1 - e \in Ra'$.*

We study the relation between EVNL-rings and other classes. More precisely, we prove that the classes of EVNL-rings, C-rings and TB-rings coincide with each other. The following theorem tells us that the class of C-rings coincides with the class of EVNL-rings.

Theorem 12 *A ring R is an EVNL-ring if and only if R is a C-ring.*

Proof. Let R be an EVNL-ring and $a,\ a' \in R$ such that $a + a' = 1$. Then there exists $b \in R$ such that $y = 1 - a'b$ is regular, hence $y = ue$ for some idempotent $e \in R$ and unit $u \in R$. Put $y' = a'b$, we get $y + y' = 1$. Using an argument similar to that in [5, Proposition 4.4], it follows that $1 - e \in Ry' = Ra'b = Rba' \subseteq Ra'$, and $e \in Ry = R(1 - a'b) \subseteq R(1 - a') = Ra$. Hence R is a C-ring.

On the other hand, assume that R is a C-ring, then for every $a \in R$, there exists $e = e^2 \in R$ such that $e \in Ra$ and $e' = 1 - e \in Ra'$ where $a' = 1 - a \in R$. Therefore $e = ab$ for some $b \in R$ and $1 - e = 1 - ab$ is a regular element, which completes the proof. ■

Definition 13 *[5, Definition 2.1] A ring R is said to be a topological boolean ring (TB-ring) if for any two distinct maximal ideals M and M' of R there exists an idempotent element e of R such that $e \in M$ and $e \notin M'$.*

In [5, Theorem 3.2.], it was shown that the class of C-rings coincides with the class of TB-rings.

Corollary 14 *A ring R is an EVNL-ring if and only if R is a TB-ring.*

It was shown in [5, Remark 2.2.] that a TB-ring is a pm-ring, but the converse need not be true in general. It was mentioned in [5, Remark 2.3.] that the converse is true in case of semilocal rings.

Corollary 15 *An EVNL-ring R is a pm-ring. Moreover, the converse is true if R is semilocal*

The converse is not true in general, for example, the ring of real valued continuous functions $A = (\mathbb{R}, \mathbb{R})$ is a pm-ring which is not an EVNL-ring see [5, Remark 2.2.].

Corollary 16 *$\mathbb{Z}_n$ is a C-ring (TB-ring) for all integers $n \geq 2$.*

Corollary 17 *The ring of formal power series $\mathbb{Z}_n[[x]]$ is a C-ring (TB-ring) for all integers $n \geq 2$.*

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